Informal Affiliated Auctions with Costly Entry

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Abstract

We analyze an affiliated first-price auction with costly participation and an unknown number of competing bidders. We contrast the symmetric equilibria of such *informal auctions* with the well-understood symmetric equilibria of formal auctions where the number of entrants is common knowledge at the bidding stage. With endogenous entry, the informal auction often yields a higher expected revenue to the seller than formal auctions. JEL CLASSIFICATION: D44

1   Introduction

In many markets, potential trading partners are dispersed and hard to identify ex-ante. An important function of the trading mechanism is to bring the different parties together so that gains from trade can be realized. In an internet auction, the pool of potential bidders may be huge even if the number of actual submitted bids is expected to be small. Such considerations arise in other contexts too. An employer may post a vacancy in order to attract applicants from a large pool of potential employees. A procurer may organize competitive tendering in order to attract service providers. A firm looking for a buyer may signal its interest in an attempt to attract firms interested in takeover. Such situations can be broadly understood as auctions, where the seller faces a non-trivial problem of attracting the buyers. How should competitive bidding be organized in such a case?

Entering an auction often entails significant costs for the buyer. In addition to the opportunity cost of time and effort spent on being physically present at an auction site, the preparation of bids may be costly as a result of concerns for due diligence. If entry costs

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are sunk at a stage where the eventual number of participants is unknown, it is natural to consider the performance of auctions where the number of bidders is random and endogenous. We call such trading mechanisms *informal auctions* to distinguish from traditional auctions where the set of bidders is known at the time of placing the bids.

We show that accounting for the uncertainty in the number of actual bidders leads to new types of bidding equilibria and to new revenue results in single object auctions with affiliated common values. In particular, we demonstrate that informal first-price auctions perform better than their formal counterparts when entry costs are significant.

The previous literature on auctions with costly participation has focused on what we call *formal auctions.* Entry decisions are taken in the first stage. In the second stage, all entering bidders see the realized number of participants as if arriving at an auction house, and bid optimally in the ensuing auction.

In contrast, each individual bidder in an informal auction knows only the equilibrium entry strategy of other potential participants but not the realized number of bidders at the time of choosing her bid. This auction format is meant to capture open selling procedures where interested bidders are invited to submit bids and the highest bidder wins and pays her own bid.

In our model, the potential bidders have observed a signal on the value of the object when deciding whether to enter or not. With correlated values, different types of bidders perceive the value of the good and their competitive environment differently. By affiliation, bidders with higher signals are intrinsically more optimistic about the value of the good. Similarly by affiliation, they believe that other bidders are more likely to be optimistic. If bids are increasing in signal, this means that bidders with high signals face more aggressive bidding from other bidders.

Of course, these two effects are present in auctions without entry costs as well. With entry costs, the second effect becomes more important. If the cost is sunk before the auction takes place, losing to competing bidders entails a cost even when not winning the auction. We show that this may lead to non-monotonic entry behavior in the sense that optimistic and pessimistic bidders may both enter and get the same ex-ante expected payoff in the auction. We also show that uncertainty and in particular different beliefs about the number of opponents gives rise to non-monotonic bidding in the sense that participating bidders with high types sometimes lose to lower types.

We concentrate on symmetric equilibria of informal auctions. We justify this choice by our focus on environments, where the set of potential bidders is not well known in advance. In an internet auction, there is no way to identify ex-ante who the truly interested potential bidders

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are. In such a situation, it is natural to consider a mechanism that treats symmetrically all the participants that it can attract. Any mechanism based on eliciting all bidders' types fails as soon as there is even a negligible cost for contacting an individual potential bidder.

The analysis of informal affiliated auctions is hard because in contrast to formal auctions, equilibria in monotone strategies may fail to exist.[1] Without monotonicity, it is not clear how one should proceed in the standard affiliated model with a continuum of signals for each bidder as in Milgrom and Weber (1982). In fact, Lauermann and Speit (2019) show that for exogenous entry rates, equilibria may fail to exist altogether.

In order to make progress, we consider a mineral rights model where the unknown common value of the object is a binary random variable. The signals are drawn from a finite set and they are assumed to be independent across the potential bidders conditional on the true value of the object. In an informal auction, each bidder is uncertain about the number of competing bidders. With correlated signals, different types of bidders have different beliefs on the realized numbers of their competitors and their signals.

One key feature of our equilibrium construction is that only a limited number of bidder types enter the auction in equilibrium. We base our analysis on the bidders' expected payoffs conditional on the state. With a binary state and affiliated signals, the beliefs of the potential bidders on the state of the world are monotone in their signals. Since we assume a large number of potential bidders, expected winnings at the auction stage must equal the entry cost for any entering type in any equilibrium of the game. If the expected payoffs in the two states differ at a fixed bid, only those potential bidders that assign the highest probability to the more favorable state make that bid. This allows us to restrict attention to models where only the bidders with the most extreme beliefs enter the auction.

We start our analysis with an example that has only two potential bidders and we demonstrate a failure of the usual linkage principle. We trace this failure to the combined effect of affiliation and uncertainty on the number of participants. This underlies our finding that the informal auction is often superior to the formal auction in terms of efficiency and expected revenue.

To see this in context, recall that Milgrom and Weber (1982) demonstrate the revenue superiority of the formal second-price auction over formal first-price auction for affiliated common value auctions. This result together with a set of results demonstrating the good revenue properties of public information disclosure are known as the linkage principle. The key idea is that for a fixed own bid, any auction format that increases the positive linkage

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[1] Landsberger (2007) is an early paper showing that existence of monotone equilibria often fails in auctions with participation costs.

between own information and the other players' bids increases the expected payment.

To see how this principle operates in the presence of uncertainty on the number of bidders, suppose that there are either one bidder (no competition) or two bidders (competition). Each bidder has either a low or a high signal on the value of the object. In a formal auction, a bidder first observes whether another bidder is present before placing bid. Consider a bidder, who makes a bid that wins against all low-signal bidders but loses against all high-signal bidders. In this case, her payment if she wins is the highest equilibrium bid from a low-signal bidder if there is competition, or alternatively zero if there is no competition. By affiliation, it is more likely that no bidders with a low signal participate if the value of the object is high. But this means that the expected payment of the high type bidder is lower than the expected payment of the low bidder. This reverses the usual linkage between own signal and payment, and renders formal auction inferior in terms of revenue to the informal auction.

In the main body of the paper, we model the game with many bidders as a Poisson game where the type of each potential player is positively correlated with the value of the object. The number of entrants of each type is drawn from an endogenously determined Poisson distribution with a parameter that depends on the true binary value of the object. An equilibrium of this game is a distribution of bids such that all entrants can cover their cost by their expected profit and no potential bidder of either type can make a positive profit by entering.

With more than two potential bidders, some additional issues arise. The first concern is that informal auctions may potentially possess multiple equilibria. If the winner of an auction is tied with other bidders with positive probability, information conditional on winning the object reflects the rationing amongst the highest bidders. This is not accounted for in the standard conditioning events of auctions with atomless bid distributions.[2] We show that with endogenous entry, the informal auction cannot have atoms. The second concern is that even the existence of a bidding equilibrium with a random number of bidders is not obvious.[3] We show that when entry distribution is endogenously given, a bidding equilibrium exists. These findings allow us to conclude that a unique symmetric equilibrium outcome in the overall game determining both entry and bids exists.

When the number of bidders is potentially large, a second linkage effect emerges, this time operating in the usual direction. The price paid in the formal auction is determined

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[2]Pesendorfer and Swinkels (1997) point out this possibility in the case of formal auctions and Lauermann and Wolinsky (2015) discuss this issue in a first-price auction with an unknown number of bidders.

[3]Lauermann and Speit (2019) in fact show that a first-price auction with an exogenously given probability distribution for the number of participants may fail to have an equilibrium.

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by the realized number of participating bidders. For a formal auction with $n-1$ other participants, the equilibrium bid by the low-signal bidders is equal to the value of the object conditional on n low signals. Due to affiliation, this payment is decreasing in n. Since a high-signal bidder finds it less likely that many other bidders have a low type, we see that the expected payment of the bidder is positively linked with the type and the usual linkage principle applies. A counteracting force results, as before, from the possibility that no other bidder is present giving rise to the reversed linkage effect discussed above. This latter effect is strong when entry rates are relatively low, and allows us to conclude that an informal auction dominates a formal auction whenever entry cost is sufficiently large.

We get our strongest results when the entry cost is relatively high (and so the expected number of entrants is relatively low), and the affiliation in the signals is strong. In this case, placing a zero bid is in the symmetric equilibrium support of both types of players. With atomless bidding strategies, this means that the expected payoff of each type of bidder coincides with the value of the good conditional on being the only entrant. This private benefit is also the maximum social benefit from inducing additional entry when restricted to symmetric strategies. We conclude that the symmetric equilibrium entry rates maximize social welfare in the class of symmetric entry strategies. Since we have large numbers of potential bidders, the expected payoff to bidders net of entry costs must be zero and as a result the seller receives the maximal symmetric surplus as her expected revenue. For these parameter values, we see then that the informal auction is the revenue maximizing mechanism in the class of all symmetric mechanisms.

1.1 Related Literature

Lauermann and Wolinsky (2017) analyze a model where the seller invites bidders to an auction and Lauermann and Wolinsky (2016) analyze a model of sequential search for trading partners in a similar setting. Our model is based on a different premise on the costs of identifying potential trading partners. If such costs are prohibitive, the seller must rely on trading partners' self selection into the mechanism. This leads us to analyze voluntary endogenous entry into auctions in a setting that is otherwise quite close to Lauermann and Wolinsky (2017).

Endogenous entry into auctions has been modeled in two separate frameworks. In the first, entry decisions are taken at an ex ante stage where all bidders are identical. Potential bidders learn their private information only upon paying the entry cost. Hence these models

can be though of as games with endogenous information acquisition.[4] French and McCormick (1984) gives the first analysis of an auction with an entry fee in the IPV case. Harstad (1990) and Levin and Smith (1994) analyze the affiliated interdependent values case. These papers show that due to business stealing, entry is excessive relative to social optimum. They also show that second-price auctions dominate the first-price auction in terms of expected revenue. All of these papers proceed under the assumption that the number of entering bidders is known at the moment when bids are submitted. Our informal auctions are thus not covered in these papers.

In the other strand, bidders decide on entry only after knowing their own signals. Samuleson (1985) and Stegeman (1996) are early papers in the independent private values setting where this question has been taken up. Due to revenue equivalence, comparisons across auction formats are not very interesting. To the best of our knowledge, common values auctions have not been analyzed in this setting.[5]

Finally some recent papers have analyzed common values auctions with some similarities to our paper. Pekec and Tsetlin (2008) provides an example where informal first-price auction results in a higher expected revenue than an informal second-price auction. The distribution of the bidders in that paper is somewhat extreme and not derived from entry decisions. Lauermann and Wolinsky (2017) and Lauermann and Wolinsky (2019) analyze first-price auctions where an informed seller chooses the number of bidders to invite to an auction. The bidders do not observe how many others were invited and hence the bidding stage analysis is as in our model with an exogenous entry rate. These papers do not compare revenues across different auction formats and since the distribution of entering bidders results from an optimal invitation decision by the seller, the analysis is quite different from our paper. Lauermann and Speit (2019) analyze bidding equilibria in a first-price auction with a random number of participants, but unlike in our paper the distribution is exogenously given. Atakan and Ekmekci (2014) consider a common value auction where the winner in the auction has to take an additional action after winning the auction. This leads to a non-monotonicity in the value of winning the auction that has some resemblance to the forces in our model that lead to non-monotonic entry (i.e. bidders with both types of signals enter

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[4]The equilibrium determination of information accuracy in common values auctions started with Matthews (1977) and Matthews (1984) and Persico (2000) extended this line of research to revenue comparisons for different auction formats. Since the number of bidders and equilibrium information acquisition decisions are deterministic, equilibrium bidding in these papers is still as in standard affiliated auctions models.

[5]Bhattacharya, Roberts, and Sweeting (2014) and Sweeting and Bhattacharya (2015) analyze IPV models with selective entry where entry decisions are conditioned on private information on the true valuation.

with positive probability).

Since we concentrate on symmetric equilibria of a game with a large number of potential entrants, our model has some similarities to the urn-ball models of matching. Similar to those models, our insistence on symmetric equilibria can be seen as a way of capturing a friction in the market that precludes coordinated asymmetric decisions. A recent example of such models is Kim and Kircher (2015) that studies matching with private values uncertainty. This approach has also been used in Jehiel and Lamy (2015) in a procurement auction setting with private (asymmetric) values. These models do not consider the case of common values.

The paper is structured as follows. In Section 2, we introduce some of the main ideas of the paper in a simple example with only two potential bidders. Section 3 presents the main model. Section 4 goes through some preliminary results to set the stage for our main results. Section 5 characterizes the unique equilibrium in both informal and formal auction models. Section 6 compares the expected revenues across different auction formats and Section 7 discusses the modeling assumptions in this paper. All proofs omitted from the main text are in the Appendix.

2 Revenue Comparisons with Two Potential Bidders

We start with a stylized two-player example that analyzes informal and formal auctions with a random number of participating bidders. Since our focus is to illustrate the revenue consequences of this modeling feature, we take the entry probabilities as exogenously given. In the main model, we endogenize entry.

A seller auctions a single indivisible good whose value $v(\widetilde{\omega})$ is common to both buyers, but depends on an unknown state $\widetilde{\omega} \in \{0, 1\}$. The seller's valuation for the good is commonly known to be zero. We assume that $v(1) > v(0) > 0$. Each buyer has a binary signal $\widetilde{\theta}$ on v that is i.i.d. conditional on $\widetilde{\omega}$. Let

$$\alpha := \Pr\{\widetilde{\theta} = h \,|\, \widetilde{\omega} = 1\} \;>\; \Pr\{\widetilde{\theta} = h \,|\, \widetilde{\omega} = 0\} =: \beta.$$

Let $q = \Pr\{\widetilde{\omega} = 1\}$ denote the common prior probability on state. After observing their signals, the two types of buyers have posteriors q_θ for $\theta \in \{h, l\}$ given by:

$$q_h = \frac{\alpha q}{\alpha q + \beta (1 - q)} > q_l = \frac{(1 - \alpha) q}{(1 - \alpha) q + (1 - \beta) (1 - q)}.$$

The non-standard feature in this example is that the bidders are uncertain about the number of bidders that are active in the auction. Each of the two *potential* bidders $i = 1, 2$

participates in the auction with probability $\pi \in (0,1)$. We assume these probabilities to be independent. Our goals is to compare the expected revenues in two auction formats that differ in their timing of bids.

In a *formal auction*, each bidder learns whether there is a competing bidder before placing her bid. If a single bidder is present at the auction, she gets the object for free. If two bidders are present, they both place bids and the higher bidder wins and pays her own bid. In an *informal auction*, the bidders place their bids before learning whether the other bidder is present or not, and the bidder whose bid is the highest wins and pays her own bid.

The formal auction with two bidders has a unique equilibrium, where bidders with low signals pool on bid
$$\underline{p} = E\left[v\left(\widetilde{\omega}\right) \Big| \widetilde{\theta}_1 = \widetilde{\theta}_2 = l \right],$$
and bidders with high signals mix on an interval $\left(\underline{p}, \overline{p}\right)$ for some $\overline{p} > \underline{p}$.

To get a sense of how uncertainty on the number of bidders and affiliated values affect revenues, it is useful to investigate how the linkage principle highlighted in Milgrom and Weber (1982) works in this context. The linkage principle suggests that auction formats that induce a positive linkage between a bidder's own signal and her expected payment conditional on winning the auction result in high expected revenues. Intuitively, a positive linkage reduces the information rents of bidders with high signals. This example shows that uncertainty on the number of participating bidders can reverse the usual linkage, i.e. make expected payment lower for the high type.

To make the comparison of expected payments across signal types meaningful in our binary-signal case, we need to find a bidding strategy that is (nearly) optimal for both types of bidders. To this effect, consider the following strategy in the formal auction: bid 0 if no other bidder is present, and bid $\underline{p} + \varepsilon$ with ε very small if the other bidder is present. The limit payoff from this strategy as ε converges to zero is the same as the payoff from the equilibrium strategy.[6] With this modified strategy, a bidder wins if the other bidder either has a low signal or does not participate. The expected payment conditional on winning and conditional on own signal θ is then

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\begin{aligned}
\mathbb{E}\left(p \,|\, \text{win}, \theta\right) \;=\; & \Pr\left(\text{other bidder does not participate} \,|\, \text{win}, \theta\right) \cdot 0 \\
& + \Pr\left(\text{other bidder participates and has low signal} \,|\, \text{win}, \theta\right) \cdot \underline{p}.
\end{aligned}
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We see here that the expected payment is decreasing in the perceived probability that the other bidder does not participate (conditional on the event that the first bidder wins).

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[6]The only difference to bidding $\underline{p}$ is that it wins tie-breaking for sure if the other bidder has a low signal and bids exactly $\underline{p}$

To see how bidders with different signals types perceive this probability, it is easiest to first consider the probability conditional on state ω. Recall that the probability that the other bidder does not participate is $1 - \pi$ irrespective of state. The probability of winning with the above mentioned strategy depends on state ω and is given by

$$\begin{aligned} \Pr\left(\text{win}\,|\,\omega = 1\right) &= 1 - \pi + \pi\left(1 - \alpha\right), \\ \Pr\left(\text{win}\,|\,\omega = 0\right) &= 1 - \pi + \pi\left(1 - \beta\right). \end{aligned}$$

By Bayes' rule, we then have

$$\begin{aligned} \Pr\left(\text{other bidder does not participate}\,|\,\text{win},\omega = 1\right) &= \frac{1 - \pi}{1 - \pi + \pi\left(1 - \alpha\right)} \\ &> \\ \Pr\left(\text{other bidder does not participate}\,|\,\text{win},\omega = 0\right) &= \frac{1 - \pi}{1 - \pi + \pi\left(1 - \beta\right)}. \end{aligned}$$

Since the high type finds it more likely that $\omega = 1$ (i.e. $q_h > q_l$), it follows that

$$\mathbb{E}\left(p\,|\,\text{win},\,\theta = h\right) < \mathbb{E}\left(p\,|\,\text{win},\,\theta = l\right).$$

Here we see a reversal of the usual linkage principle: for the same bidding strategy, the high type expects to pay strictly less than the low type conditional on winning.

Consider now the informal auction where the bids are made before learning whether the other bidder is present or not. It is easy to show by standard arguments that symmetric equilibrium bidding strategies in such an auction cannot have atoms as long as $\pi < 1$. Assume for a moment that the auction has a symmetric equilibrium with non-overlapping bidding supports: the low type bidders mix on some interval $(0, p')$ and the high types mix on interval (p', p'') with $p'' > p'$.

The constant strategy of bidding p' regardless of type (which is in the supports of both types) induces the same allocation as the hypothetical bidding strategy that we used above to compute the expected payments in the formal auction. Since the bidding decision is now made ex-ante, there is no linkage between own signal and expected payment conditional on winning: payment is exactly p' for both types. For the seller, no linkage between signal type and expected payment is better than a negative linkage, and it follows that the informal auction gives a higher expected revenue to the seller than the formal auction.

We want to highlight another feature of the affiliated model. The strategy profile with non-overlapping supports described above constitutes an equilibrium only for some parameter values. When such equilibria do not exist, the difference between the two auction formats is even more striking. If affiliation is relatively strong (i.e. $\alpha/\beta >> 1$), if there is substantial

uncertainty on participation ($\pi \ll 1$), and if the valuation is not too sensitive to state ($v(1)/v(0)$ is relatively close to 1), equilibria with non-overlapping supports do not exist.[7] Intuitively, with highly correlated signals, high-type bidders do not expect competition from low-type bidders and they are tempted to reduce their bid to zero. We show in the analysis of the main model that in this case, the informal auction has a unique equilibrium where the bidding supports of both types of bidders include 0. This implies that the rent of the high type bidder is below the rent that she would get in the formal auction and consequently, the revenue to the seller is higher. As we will show in detail later in this paper, this has an important qualitative implication when considering endogenous entry: both types of bidders will be exactly rewarded by the contribution that they make to the social surplus, and this leads to efficient entry.

In this example, a positive reserve price below the reservation value of the most pessimistic bidder would raise the expected revenue in the model with exogenous entry. This is because we have treated entry as exogenous. If entry is endogenous, the bidders must recover their entry costs at the bidding stage. In our main model, we show that a reserve price hurts the low-type bidders and distorts equilibrium entry in a particularly detrimental way and results in a lower expected revenue to the seller.

3   Model

A seller has a single indivisible object for sale. The value of the object, $v(\widetilde{\omega})$, is common to all potential bidders, and this value depends on unknown state $\widetilde{\omega} \in \{0, 1\}$. We assume $v(1) > v(0) > 0$. We normalize the value to the seller to be zero so that $v(\widetilde{\omega})$ is also the social surplus generated by trade.

The number of bidders that participate in the auction is a Poisson random variable with an *endogenously* given parameter. Since the seminal work of Myerson on games with uncertain numbers of participants, the Poisson game model has been widely used in models of information economics.[8] It is a tractable model that allows the random number of participating bidders to be correlated with the state of the world (the true value of the object). As

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[7]The exact condition for non-overlapping bid supports in equilibrium is

$$\frac{(1 - \alpha) v(1)}{1 - \alpha \pi} \geq \frac{(1 - \beta) v(0)}{1 - \beta \pi}.$$

[8]For a discussion of the Poisson entry model in the context of a procurement auction with private values, see Jehiel and Lamy (2015).

in the previous section with two potential participants, this correlation can be rationalized by entry decisions based on affiliated signals.

To see how one arrives naturally at the Poisson model, assume for the moment that there are N potential entrants that observe signals $\widetilde{\theta} \in \{\theta_1, ..., \theta_M\}$ with a conditional distribution

$$\alpha_{\omega,m} =: \Pr\{\widetilde{\theta} = \theta_m \,|\widetilde{\omega} = \omega\} \text{ for } m \in \{1, ..., M\},$$

where state is $\omega \in \{0, 1\}$. We label the signals so that they satisfy strict monotone likelihood ratio property: $\frac{\alpha_{1,m}}{\alpha_{0,m}}$ is increasing in m. If each entrant with signal m enters with probability $\pi_{m,N}$, then the number of type m entrants is a binomial random variable with parameters $(\alpha_{1,m}\pi_{m,N}, N)$ and $(\alpha_{0,m}\pi_{m,N}, N)$ in states $\omega = 1$ and $\omega = 0$, respectively. Keeping the expected number of entrants $\alpha_{\omega,m}\pi_{m,N} \cdot N \to \alpha_{\omega,m}\pi_m$ constant, let N increase towards infinity and consider the binomial variables $\mathrm{Bin}(\alpha_{\omega,m}\pi_{m,N}, N)$. The limiting random number N_ω^m of entrants of type m in state ω has then a Poisson distribution with parameter $\lambda_{\omega,m} := \alpha_{\omega,m}\pi_m$. The random variables N_ω^m are furthermore independent. We use the notation N^m to denote the total random number of participating bidders of type m without conditioning on state.

Motivated by this limiting argument, we model the auction directly as a Poisson game where π_m are endogenously determined parameters. Each potential bidder perceives the number of other participants of type m to be given by a Poisson random variable N_ω^m with parameter $\lambda_{\omega,m} = \alpha_{\omega,m}\pi_m$ that depends on π_m and the distribution of the signal $\widetilde{\theta}$. In the *informal* auction, the participating bidders place their bids before learning the realized number of other participating bidders. The bidder with the highest bid wins and pays her own bid. A symmetric strategy profile is a pair (b, π) where $b : \{1, ..., M\} \to \Delta(\mathbb{R}_+)$ is a bidding strategy for each type of participating bidders and $\pi : \{1, ..., M\} \to \mathbb{R}_+$ is a vector of entry rates. By a symmetric equilibrium of the informal auction, we mean a pair (b, π) such that b is a bidding equilibrium given π, and no type m gets a higher expected payoff than c at the auction stage. If $\pi_m > 0$, then the expected payoff of type m is exactly c.

We will later compare the informal auction with a *formal* (first-price) auction where the bidders learn the number of competing bidders before placing their bids. In the formal auction, the entry strategy is as above, but the bidding strategy $b : \{1, ..., M\} \times \mathbb{N} \to \Delta(\mathbb{R}_+)$ conditions on the realized number of participating bidders. As with the informal auction, the bidder with the highest bid wins and pays her own bid.

4   Preliminary Analysis of the Informal Auction

In this section, we derive a number of preliminary results. First, we derive the symmetric social planner's entry profile as a benchmark and show that no other type but θ_1 and θ_M enter with positive probability. This result holds also for all symmetric equilibria of the informal auction. We also show that the bidders must use atomless bidding strategies. Finally, we fix an exogenous entry profile by the extreme types and derive the corresponding atomless bidding equilibrium. We make use of these results in the next section when we characterize the full equilibrium with endogenous entry.

4.1   Symmetric Planner's Optimum

We start by analyzing the symmetric benchmark where the social planner chooses $\pi = (\pi_1, ..., \pi_M)$ to maximize the expected gain from allocating the object net of the expected entry costs. The planner's objective is to

$$\max_{\pi \geq 0} W(\pi),$$

where $W(\pi)$ is the expected total surplus with a given entry profile:

$$W(\pi) = q[v(1)\left(1 - e^{-\Sigma_m \alpha_{1,m}\pi_m}\right) - c\Sigma_m \alpha_{1,m}\pi_m]$$
$$+ (1 - q)\left[v(0)\left(1 - e^{-\Sigma_m \alpha_{0,m}\pi_m}\right) - c\Sigma_m \alpha_{0,m}\pi_m\right].$$

The first order conditions for interior solutions to this concave problem are given by:

$$q_m v(1) e^{-\Sigma_m \alpha_{1,m}\pi_m} + (1 - q_m) v(0) e^{-\Sigma_m \alpha_{0,m}\pi_m} = c,$$

for all m, where q_m is the belief of type θ_m on state:

$$q_m = \frac{\alpha_{1,m}q}{\alpha_{1,m}q + \alpha_{0,m}(1-q)}.$$

Its unique solution is given by:

$$v(1) e^{-\Sigma_m \alpha_{1,m}\pi_m} = c, \tag{1}$$
$$v(0) e^{-\Sigma_m \alpha_{0,m}\pi_m} = c.$$

Taking logarithms, we have

$$\Sigma_m \alpha_{1,m}\pi_m = \log\left(\frac{v(1)}{c}\right),$$
$$\Sigma_m \alpha_{0,m}\pi_m = \log\left(\frac{v(0)}{c}\right).$$

Since we have two linear equations in M variables, the solutions to this pair of equations are not unique. By affiliation, we have $\frac{\alpha_{1,M}}{\alpha_{0,M}} \geq \frac{\alpha_{1,m}}{\alpha_{0,m}} \geq \frac{\alpha_{1,1}}{\alpha_{0,1}}$. Hence we have a positive solution for $\pi = (\pi_1, ..., \pi_M)$ only if

$$\alpha_{1,1}\pi_1 + \alpha_{1,M}\pi_M = \log\left(\frac{v(1)}{c}\right),$$

$$\alpha_{0,1}\pi_1 + \alpha_{0,M}\pi_M = \log\left(\frac{v(0)}{c}\right),$$

has a positive solution. Again by affiliation, we see that this is the case only if

$$\alpha_{0,M}\log\left(\frac{v(1)}{c}\right) < \alpha_{1,M}\log\left(\frac{v(0)}{c}\right).$$

If this condition is satisfied, a solution $(\pi_1, 0, ..., 0, \pi_M)$ with $\pi_M > \pi_1 > 0$ to this system exists. Other solutions to this problem exist, but the aggregate amount of entry is uniquely determined for both states across all such solutions.

We can compute a threshold $\bar{c}$ such that positive entry by multiple types takes place if the entry cost is below $\bar{c}$:

$$\bar{c} = e^{\frac{\alpha_{1,M}\log(v(0)) - \alpha_{0,M}\log(v(1))}{\alpha_{1,M} - \alpha_{0,M}}} > 0.$$

For $c \in (0, \bar{c})$ we have an interior solution with $\pi_M^{opt} > \pi_1^{opt} > 0$.

When $c \geq \bar{c}$, we have a corner solution where $\pi_m^{opt} = 0$ for all $m < M$. In that case, the optimal entry rate for the θ_M types is solved from

$$q_M v(1) e^{-\alpha_{1,M}\pi_M} + (1 - q_M) v(0) e^{-\alpha_{0,M}\pi_M} = c.$$

This gives an interior solution $\pi_M^{opt} > 0$ as long as $c < \bar{\bar{c}}$, where

$$\bar{\bar{c}} = q_M v(1) + (1 - q_M) v(0).$$

To summarize, for low entry cost $c < \bar{c}$ the socially optimal entry profile features the two extreme types entering with positive rate, for intermediate entry cost $c \in [\bar{c}, \bar{\bar{c}})$ only highest type bidders enter with positive rate, and for high entry cost $c \geq \bar{\bar{c}}$ there is no entry. Note that the threshold $\bar{\bar{c}}$ is the expected value of the object for a player that has observed the highest signal. This is intuitive: as long as entry cost c is below the expected value of the object for the most optimistic potential entrant, it is socially optimal to have at least some entry.

4.2 No Pooling with Endogenous Entry

In any entry equilibrium, entrants earn an expected profit of c at the bidding stage. Hence for all bids p in the union of the bid supports of all bidders, the equilibrium payoff of any bidder submitting p must be c and the payoff for any other type cannot exceed c since otherwise we would have a profitable deviation for some individual potential entrant. In this section, we use this property to rule out the possibility of pooling bids (i.e. bids chosen with a strictly positive probability by some bidder type).

With a random number of bidders, the analysis of pooling bids is more delicate than in the case of a fixed number of bidders. With uniform tie-breaking for highest bids, a bidder submitting a pooling bid is more likely to win if the number of tying bids is small. The number of tying bids contains information on the realized number of participating bidders of different types. Since the distribution of different types depends on state, such information is in turn informative on the state of the world and hence on the value of the object. The additional information contained in the event of winning the auction must be accounted for when calculating the optimal bid. Lauermann and Wolinsky (2017) show that such rationing effects may indeed lead to bidding equilibria with pooling.

We now argue that this cannot occur in our model with endogenous entry. As observed by Pesendorfer and Swinkels (1997), it is not possible to have pooling bids that are made only by the highest types. If they were the only bidders to pool on a bid p, then the rationing effect would be negative: winning at bid p is more likely when there are few tied bidders. But if only high type bidders bid p, then winning with a tied bid decreases the posterior on $\{\omega = 1\}$ and the value of the object conditional on winning is lower than the value conditional on the event that the bidder is tied for the highest bid. By bidding $p + \varepsilon$, for a small enough ε, the bidder wins in the event of a tied bid without any rationing and makes a positive gain from the deviation. As a result, pooling by high types only is not possible.

To rule out other types of pooling bids, we note first that there cannot be pooling at any $p < v(0)$. This is because for such a low bid, winning is profitable in both states and hence a small upward deviation would be strictly profitable. With equilibrium entry, only the highest type θ_M can bid above $v(0)$. To see this, note that winning with a bid $p \geq v(0)$ is profitable only if $\omega = 1$. By affiliation, $\Pr\{\omega = 1 | \theta_m\}$ is increasing in m. Therefore, if θ_m gets in expectation c by bidding $p \geq v(0)$, then type θ_M gets strictly more than c by bidding p, which is not compatible with interim entry equilibrium. Since it is not possible to have pooling bids that are made only by high types, pooling is not possible for any $p \geq v(0)$ either. We have hence proved:

Lemma 1 *There are no atoms in the bidding distribution b of a symmetric equilibrium (b, π).*

4.3 Entry by Extreme Types

We consider next the types of bidders that can enter in equilibria with atomless bidding strategies at the auction stage. Denote by $R_\omega(p)$ the expected rent at the auction stage in state ω from bid p:

$$R_\omega(p) = (v(\omega) - p) \prod_{m=1}^{M} e^{-\lambda_{\omega,m}(1 - F_m(p))},$$

where $F_m(p)$ is the bid distribution of type θ_m.

For each p, there are three possibilities: either i) $R_1(p) = R_0(p)$, ii) $R_1(p) > R_0(p)$, or iii) $R_1(p) < R_0(p)$. All types of bidders are indifferent between entering and submitting a bid p in case i). In case ii), only type θ_M can make bid p in equilibrium with interior entry probabilities. If θ_m, $m < M$, makes the bid, her expected payoff at the bidding stage is $R_0(p) + q_m(R_1(p) - R_0(p))$ and she must earn at least c to be willing to enter and bid p. But since $q_m < q_M$, type θ_M will get strictly more than c by entering and bidding p, which cannot be an equilibrium. In case iii), we have similarly that only type θ_1 can enter in such an equilibrium.

Our equilibrium construction shows that all the cases i) - iii) can occur in equilibrium, but the bid distribution is fully pinned down by the parameters of the bidders that bid in cases ii) and iii). The only remaining indeterminacy concerns the exact composition of bidders that make bids where case i) holds. But the requirement that the expected payoffs be equalized across the two states determines the aggregate distributions of bids for each state in a manner completely analogous to the payoff equalization in the planner's problem across the two states. As in the planner's case, these aggregate distributions can be generated with positive entry by types θ_1 and θ_M. We formalize this discussion in the following Lemma:

Lemma 2 *Let $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ denote an equilibrium of a reduced model, where only types $\{\theta_1, \theta_M\}$ exist. Then $\left(\bar{b}, (\bar{\pi}_1, 0, ..., 0, \bar{\pi}_M)\right)$ remains an equilibrium of the full model that allows entry by all types $\{\theta_1, ..., \theta_M\}$. Moreover, if there is another equilibrium (b, π) in the full model, this equilibrium is equivalent to an equilibrium of the reduced model in terms of induced probability distribution of bids and revenues.*

4.4 Bidding Equilibrium of the Two-type Model

Based on Lemmas 1 and 2 we can restrict our attention to atomless bidding equilibria of a model with binary bidder types $\theta \in \{l, h\}$, where labels l and h correspond to the extreme

types θ_1 and θ_M of the full model, respectively. In this subsection we further characterize the set of possible bidding equilibria b. Since b must be a best response given the entry rate profile $\pi = (\pi_h, \pi_l)$ for both types, we can treat π here as exogenously given. The full equilibrium with endogenous entry will be analyzed in the next section.

The number of bidders with signal θ in state ω is a Poisson random variable N_ω^θ, where the parameter $\lambda_{\omega,\theta}$ depends on π_θ through

$$\lambda_{\omega,\theta} = \alpha_{\omega,\theta} \cdot \pi_\theta.$$

With affiliated signals, $\alpha_{1,h} > \alpha_{0,h}$ and $\alpha_{1,l} < \alpha_{0,l}$. This in turn implies that $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{1,l} < \lambda_{0,l}$ for any entry profile $\pi = (\pi_h, \pi_l)$. The expected payoff from a bid p to a bidder of type θ when the other players bid according to the profile $\mathbf{F} = (F_h(\cdot), F_l(\cdot))$ is:

$$
\begin{aligned}
U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))} (v(1) - p) \\
&\quad + (1 - q_\theta) e^{-\lambda_{0,h}(1-F_h(p))} e^{-\lambda_{0,l}(1-F_l(p))} (v(0) - p).
\end{aligned}
$$

A symmetric bidding equilibrium for entry rate profile π is a bid profile $\mathbf{F}$ such that for all $p \in \mathrm{supp} F_\theta(\cdot)$, p maximizes $U_\theta(p)$. Using this property and the functional form of $U_\theta(p)$, we can pin down some key properties of a bidding equilibrium. Lemma 3 below shows that depending on the entry rate profile, we are in one of two qualitatively distinct cases. The former case is just a discrete type version of a standard monotonic equilibrium: a higher type always bids higher than a low type. The latter case is more special and shows that bidding equilibrium may fail to be monotonic: the bid supports overlap and hence a low type wins against a high type with a positive probability.

Lemma 3 *Let b be an atomless bidding equilibrium for given entry rate profile $\pi = (\pi_h, \pi_l)$. Then the union of the bid supports of the two types is an interval and the bid support of the low type is an interval that contains 0. If*

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} < \frac{v(1)}{v(0)},$$

then the bid supports are non-overlapping intervals with a single point in common. If

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)},$$

then the bid supports intersect and 0 is contained in the supports of both types.

It should be emphasized that we have not yet proven the existence of an equilibrium to the model. The Lemmas above should therefore be understood as a characterization of a candidate for an equilibrium bidding strategy. Lauermann and Speit (2019) show that a bidding equilibrium may fail to exist in a first-price auction with an exogenously given probability distribution for the participants. Also in our setup, an atomless bidding equilibrium may fail to exist if entry rates are exogenously given.[9] In the next section we show that with endogenous entry rates our model has always a unique symmetric bidding equilibrium and so the characterization in Lemma 3 is valid.

5   Equilibrium with Endogenous Entry

In this section, we consider the equilibrium entry rates in both informal and formal auction formats. We analyze their properties through a comparison with the social planner's solution. Therefore, we set the stage in subsection 5.1 by explaining the distinction between private and social entry incentives. In subsection 5.2 we analyze the equilibrium of the informal auction and in subsection 5.3 we analyze the equilibrium of the formal auction.

5.1   Private and Social Incentives for Entry

It is helpful to start by considering a purely hypothetical game where the object is given for free to the entrant if there is a single entrant, but with two or more entrants the object is withdrawn from the market and the entrants forgo the entry cost. Denote by $V_\theta^0(\pi_h, \pi_l)$ the expected value of a bidder of type $\theta \in \{h, l\}$ at the auction stage when the symmetric entry profile is (π_h, π_l):

$$
\begin{aligned}
V_h^0(\pi_h, \pi_l) &= q_h e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l} v(1) + (1 - q_h) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l} v(0), \\
V_l^0(\pi_h, \pi_l) &= q_l e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l} v(1) + (1 - q_l) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l} v(0).
\end{aligned}
$$

Any (π_h, π_l) such that $V_h^0(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l) = c$ in the above equations guarantees that every potential entrant is indifferent between entering and staying out. By inspection, we see that this is the same condition as in (1), and hence private entry incentives coincide with social incentives. To understand why this is the case, note that in this hypothetical situation each entrant gets exactly her marginal contribution to the social welfare as her payoff. Since the value of the object is common to all the players, an entrant contributes to the total surplus if and only if she is the only entrant. Since the object is given free in such

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[9]We thank Andrea Speit for pointing this out to us.

a situation her private benefit equals her marginal contribution. Therefore, entry incentives coincide exactly with the social value of entry. As we will show below, the different auction formats give (at least weakly) too high entry incentives for the high types. Quantifying this excess incentive across different auction formats gives us our revenue comparisons.

In Figure 1, we illustrate the social optimum in the case with interior entry probabilities by drawing the planner's reaction curves

$$\pi_l^* (\pi_h) \; : \; = \arg\max_{\pi_l \geq 0} W (\pi_h, \pi_l) ,$$
$$\pi_h^* (\pi_l) \; : \; = \arg\max_{\pi_h \geq 0} W (\pi_h, \pi_l) ,$$

in the $(\pi_h, \pi_l)-$plane. The social optimum $\left(\pi_h^{opt}, \pi_l^{opt}\right)$ is at the intersection of these curves. As shown above, these curves are also the indifference contours $V_h^0 (\pi_h, \pi_l) = c$ and $V_l^0 (\pi_h, \pi_l) = c$ for a potential entrant of type h and l, respectively, who gets her marginal contribution as expected payoff.

Auctions often give an additional reward to high type entrants on top of their social contribution. This distorts the entry rates from socially optimal levels. Since bidding zero is in the support of the low type bidders by Lemma 3, their equilibrium entry incentives coincide with the planner's incentives conditional on the entry rate of the high types. As a result, their equilibrium reaction curve coincides with the planner's reaction curve and hence any equilibrium entry profile must be located along the planner's reaction curve $\pi_l^* (\pi_h)$ for the low type. For our revenue comparisons it is therefore useful to analyze how movements along that curve change the total surplus.

We denote by $W^* (\pi_h)$ the total surplus as a function of the high types' entry rate, when low types adjust entry optimally:

$$W^* (\pi_h) := \max_{\pi_l \geq 0} W (\pi_l, \pi_h) .$$

The following Lemma shows that $W^* (\pi_h)$ is single peaked in π_h with its peak at π_h^{opt}. This is illustrated in Figure 1 by arrows along $\pi_l^* (\pi_h)$ that point towards increasing social surplus. Based on this Lemma we can compare two auction formats simply by analyzing how far they distort the entry rate of the high type away from the social optimum.

Lemma 4 *Let $\pi_h^{opt} > 0$ denote the socially optimal high type entry rate. We have*

$$\frac{dW^* (\pi_h)}{d\pi_h} \begin{cases} > 0 \; \textit{for } \pi_h < \pi_h^{opt} \\ = 0 \; \textit{for } \pi_h = \pi_h^{opt} \\ < 0 \; \textit{for } \pi_h > \pi_h^{opt} \end{cases} . \tag{2}$$

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5.2 Entry Equilibrium in Informal Auctions

In this subsection we will present our main result, which states that the informal first-price auction with endogenous entry has an essentially unique symmetric equilibrium. We use the three Lemmas of the previous Section to narrow down the set of possible equilibria in various ways. By Lemma 1 there are no symmetric equilibria with pooling in the bidding stage, so we can concentrate on atomless bidding strategies. By Lemma 2 we can without loss concentrate on the lowest and highest type of potential entrants. Finally, Lemma 3 further narrows down the form of bidding strategies to two distinct cases. In this section we show that a unique equilibrium satisfying these Lemmas exists.

Consider first the case where $c \geq \bar{c}$ and the planner's solution has $\pi_l^{opt} = 0$. Suppose that this is the case also in equilibrium. Since all the entering bidders have observed a high signal, it is easy to see that in both auction formats that we consider, the expected payoff to the entering bidders is given by the probability that no other bidder entered times the expected value of the object conditional on that event. Since the updated belief of a high type bidder on $\{\omega = 1\}$ is given by $q_h = \frac{q\alpha}{q\alpha + (1-q)\beta}$, equating the expected benefit from entry to the cost of entry gives:

$$q_h e^{-\alpha \pi_h} v(1) + (1 - q_h) e^{-\beta \pi_h} v(0) = c.$$

This coincides with the planner's optimality condition. Hence the symmetric equilibrium entry profile in both auction formats coincides with the planner's optimal solution. The key reason for this is the lack of heterogeneity in the bidders' information.

In equilibrium the bidders are indifferent between entering and not entering. Hence their expected payoff must be at their outside option of 0. Since the auctions generate maximal social surplus (under the restriction to symmetric entry profiles) in this case and since bidders expected payoff is at zero, the seller collects the entire expected social surplus in expected revenues. Hence both auction formats are also revenue maximizing within the class of symmetric mechanisms (where we require symmetry at the entry stage as well as at the bidding stage).

Remark 1 *If $\bar{c} \leq c < \bar{\bar{c}}$, then only the high type enters and both auction formats are efficient and hence revenue equivalent. If $c \geq \bar{\bar{c}}$, then there is no entry in the planner's solution nor in either of the auction formats.*

We now move to the more interesting case $0 < c < \bar{c}$, where the planner's solution induces entry by both types. We denote by $V_\theta^{IF}(\pi_h, \pi_l)$ the expected payoff of a bidder with type θ

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at the bidding stage of the informal first-price auction for given entry rates π_h, π_l. We show that there is a unique pair $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ such that $V_h^{IF}\left(\pi_h^{IF}, \pi_l^{IF}\right) = V_l^{IF}\left(\pi_h^{IF}, \pi_l^{IF}\right) = c$.

In addition to the uniqueness, the proposition below contains a qualitative statement about the equilibrium. A low type entrant gets a payoff in the bidding stage that is exactly equal to her social contribution:

$$V_l^{IF}\left(\pi_h, \pi_l\right) = V_l^0\left(\pi_h, \pi_l\right),$$

which means that equilibrium entry point $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ must be located along the planner's reaction curve $\pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right)$. A high type may get more, so that to in order to dissipate excess rent of the high type, we must have $\pi_h^{IF} \geq \pi_h^*\left(\pi_l^{IF}\right)$. As a result, the equilibrium entry rate must be (weakly) too high for the high type and (weakly) too low for the low type:

Proposition 1 *Let $0 < c < \bar{c}$. The informal first-price auction has a unique symmetric equilibrium where only the extreme types enter with strictly positive rate. The entry rates by the highest and lowest types, respectively, satisfy $\pi_h^{IF} \geq \pi_h^{opt}$ and $\pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right) \leq \pi_l^{opt}$. All entering bidder types use atomless bidding strategies. Zero bids are always in the low-type bid support and the upper bound of the high-type bid support is given by*

$$p^{\max} := q_h v\left(1\right) + \left(1 - q_h\right) v\left(0\right) - c.$$

Any equilibrium (b, π) with positive entry by interior types induces the same probability distribution of bids and revenues as the equilibrium described above.

5.3   Entry Equilibrium in Formal Auctions

We showed in Lemma 2 above that in the case of informal auction, we can restrict our analysis to the two-type model. The same is true for the formal auctions. In fact, we can show that the bidding equilibria of formal auctions are simpler than in the informal auction in the sense that for all bid levels, either the lowest or the highest type has a strict advantage over any intermediate type. This implies that the intermediate types can never break even in equilibrium and so there cannot be any equilibria where intermediate types enter with positive rate.

Lemma 5 *Let $\pi = \left(\pi_1, ..., \pi_M\right)$ denote the vector of entry rates by different types in an equilibrium of the formal auction. Then $\pi_2 = ... = \pi_{M-1} = 0$, i.e. only types θ_1 and θ_M may enter with a strictly positive rate.*

Since only two types enter in equilibrium, we can utilize the following characterization of the unique symmetric bidding equilibria in formal auctions taken from Chi, Murto, and Välimäki (2019).

Lemma 6 *With two bidder types and exogenous entry rates, the formal first-price auction format has a unique symmetric equilibrium. If $n = 1$, the only participant bids zero. With $n \geq 2$ participants, the low type bidders pool at bid*

$$\underline{p}(n) = E\left[v \,\middle|\, \theta = l, \, N^l = n - 1, \, N^h = 0\right]$$

and the high types have an atomless bidding support $\left[\underline{p}(n), \overline{p}(n)\right]$ with some $\overline{p}(n) > \underline{p}(n)$.

The key property for our analysis is that in a formal auction, a bidder with a high signal always earns a higher private benefit in the bidding stage than her contribution to the social surplus. To see this, recall that in a model with common values additional entry is socially valuable only if no other bidder participates in the auction. In the formal auction, the bidder with a high type gets the social benefit (i.e. object for free if $n = 1$), but she also receives an extra information rent when bidding against bidders with low signals when $n \geq 2$. By bidding just slightly above $\underline{p}(n)$, say $\underline{p}(n) + \varepsilon$, a high type wins against low types, and her payment $\underline{p}(n) + \varepsilon$ is lower than her expectation of the value of the object given her signal. As a result, a formal auction in our setup features excessive entry by the high types relative to the planner's solution, as we confirm formally below.

The bidding equilibrium characterized above leads naturally to existence and characterization of a full entry equilibrium with endogenous entry. Proposition 2 below mirrors Proposition 1 for informal auctions. The main qualitative difference is that unlike in an informal auction, the entry rates are here always strictly distorted away from the socially optimal levels for the reason outlined above (as long as entry cost is so low that both types enter).

Proposition 2 *Let $0 < c < \overline{c}$. The formal first-price auction has a symmetric equilibrium where only the extreme types enter with strictly positive rate. The equilibrium entry rates by the highest and lowest types, respectively, satisfy $\pi_h^F > \pi_h^{opt}$ and $\pi_l^F = \pi_l^*\left(\pi_h^F\right) < \pi_l^{opt}$.*

Note that if $c > \overline{c}$, then the low type does not enter and Remark 1 applies.

6   Revenue Comparisons

In this section we compare the expected revenue between informal and formal auction formats. We assume throughout this section that $c < \bar{c}$. If $c \geq \bar{c}$, then both auction formats are revenue equivalent as already stated in Remark 1.

Our first revenue comparison result shows that when affiliation of signals is strong enough and when entry cost is sufficiently large, the informal auction gives the entire (symmetric) social surplus to the seller in expected revenues. To see when exactly this happens, recall from Lemma 3 that the bid supports of both type of bidders must contain 0 whenever

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)}.$$

The left hand side is decreasing in π_l, and approaches $\alpha_{0,l}/\alpha_{1,l}$ when entry rate of the low type, π_l, becomes small. The ratio $\alpha_{0,l}/\alpha_{1,l}$ measures the difference across the states of the probability of observing a low signal, and reflects divergent beliefs of different types on the degree of competition. Hence, whenever this ratio is larger than the ratio $v(1)/v(0)$, a sufficiently high entry cost will ensure such a low entry rate for the low type bidders that high type bidders are willing to bid zero. This means that the high type bidders get no rent on top of their contribution to the social welfare in the informal auction. When this is the case, the informal first price auction maximizes the seller's expected revenue in the class of *all* symmetric mechanisms. In contrast, in the formal auctions the high type will always get an information rent that strictly exceeds her social contribution, which distorts entry rates away from social optimum. This establishes the strict superiority of the informal first price auction for this case.

Proposition 3 *If*

$$\frac{\alpha_{0,l}}{\alpha_{1,l}} > \frac{v(1)}{v(0)}, \tag{3}$$

then there is a $\widetilde{c} < \bar{c}$ such that for $c \in (\widetilde{c}, \bar{c})$, entry is efficient in the informal auction and the expected revenue is strictly higher than in the formal auction.

When (3) does not hold and/or entry cost is very low, the revenue comparison is less straightforward since also the informal auction induces too much entry by the high types. The question is then, which of the auction formats will induce less distortion, i.e. lower entry incentive to the high type. We show below that as long as the entry cost is substantial enough, the superiority of the informal auction continues to hold, but a more elaborate argument

is needed. The key to understanding this is the linkage principle under uncertainty on the number of competing bidders. We will elaborate this below.

The linkage principle as commonly understood suggests that it is a good thing for the seller if an auction format induces a positive linkage between a bidder's own signal and her expected payment, conditional on winning the auction. A rough intuition is that such a linkage will reduce the information rent of the bidder with a high signal. To understand how this principle operates in our model, let us consider the formal auction. Recall that for any realized number of bidders n, this auction format has a unique bidding equilibrium where the low types pool at $\underline{p}(n)$ and the high types mix over interval $\left[\underline{p}(n), \overline{p}(n)\right]$. Let us consider a strategy, where the bidder wins against low types but loses against high types, i.e. bid $\underline{p}(n) + \varepsilon$ with ε very small. Note that this strategy is almost a best-response to both type of bidders and therefore allows meaningful comparison between expected payments across the signal types (even though it is not within equilibrium support for the low type bidders, in the limit where ε gets very small low types are indifferent between bidding $\underline{p}(n)$ and $\underline{p}(n) + \varepsilon$; the only effect of deviating up by ε is that this will tilt the rationing to the deviator in case of a tie at $\underline{p}(n)$).

Now, consider the linkage between signal and expected payment using this strategy. The expected payment is the expectation of $\underline{p}(n)$ over n, where we define $\underline{p}(1) \equiv 0$ since a solitary bidder bids zero. By affiliation, a high type expects a smaller number of low type bidders, and hence a lower n conditional on no high types. Therefore, if $\underline{p}(n)$ were decreasing in n, the linkage principle would work in the normal way: a high type bidder expects to pay more than the low type. By affiliation, $\underline{p}(n)$ is indeed decreasing in n for $n \geq 2$ since the posterior on $\omega = 1$ is decreasing in the number of low type bidders. Importantly, however $\underline{p}(1) = 0 < \underline{p}(2)$, which breaks the monotonicity of $\underline{p}(n)$. A high type bidder evaluates the probability that there are no low type bidders present to be higher than what a low type believes, and this will reduce the expected payment of the high type relative to the low type. If the entry rate of the low type bidders is small enough, this prospect of facing no competition certainly dominates and makes the linkage between signal and expected payment negative. This case mirrors exactly the example with only two potential bidders presented in Section 2. If the entry rate of the low type is very high, we are no longer able to say which effect dominates.

In the informal first-price auction, there is no linkage between own signal and expected payment since the bidders pay their own bids whenever they win. It follows that whenever the linkage between the signal and the expected payment in the formal second-price auction is negative, the high type bidder's expected rent is higher in the formal auction

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than in the informal auction, and vice versa if the linkage is positive. As argued above, the linkage is negative whenever the possibility that no low type bidder enters is substantial enough. Translated into the exogenous parameters, this means that whenever the entry cost is high enough, the formal auction induces a higher distortion in the high-type bidder's entry incentives, and so the informal first-price auction raises more revenue.

Proposition 4 *There is some $c'' < \bar{c}$, such that for $c \in (c'', \bar{c})$, informal first-price auction generates a strictly higher expected revenue than formal auctions.*

7   Discussion

7.1   Entry Fees and Reserve Prices

One can ask if revenue can be further enhanced by other auction design elements such as entry fees or reserve prices. If the conditions in Proposition 3 hold, then the informal auction without any additional elements already induces the highest possible total surplus and gives this entirely to the seller. In that case it is clear that no symmetric mechanism can improve revenue.

If the informal auction is inefficient, then there is a distortion in entry rates that could potentially be reduced or even eliminated. The distortion results from the positive equilibrium rent of the high-type bidders. Hence it would be desirable to find an instrument that penalizes high types but not the low types. Since information is private, this does not appear to be feasible.

An entry fee is an instrument that penalizes both types equally. As a result, adding a small entry cost distorts the entry incentive of the high types in the intended direction, but at the same time affects the low types in the wrong direction. An argument based on the envelope theorem suggests that a small entry cost could marginally increase revenue. To see this, note that in equilibrium, entry by the low types is socially optimal conditional on the entry rate of the high types. Therefore, a slight change in the low type entry rate will not have a first-order effect on the total surplus. In contrast, the high type entry rate is already distorted away from social optimum and therefore a small change has a first-order effect in total surplus and revenue.

Another alternative would be to use reserve prices. Intuitively, this may seem attractive as it would increase revenue especially in a situation where only one bidder participates. However, it should be kept in mind that the rent that a bidder gets when entering alone is important in encouraging entry. All bidder rents are dissipated in the bidding stage in any

case, and hence any effects on revenue should be evaluated based on their effect on total surplus. From this angle we see that the reserve price is actually a worse instrument than an entry fee since it penalizes the low types more than the high types. To see this, note that a reserve price reduces most heavily the bidder rents in situations where there is only one bidder present. In equilibrium the low type breaks even by counting on the event that no other bidder enters, whereas the high type counts relatively more on the value being high.

This can be seen most clearly by considering a reserve price set above $v(0)$. This implies that entering the auction can be profitable only in state $\omega = 1$. But since the highest type M always has the highest posterior on state $\omega = 1$, no other type can break even if type M does, and as a result all the lower types are completely eliminated from the market. For the reasons that we have outlined in the paper, this could never be optimal for the seller if $c < \bar{c}$.

7.2   Second-Price Auctions

We view the first-price auction format as the natural model to describe bidding contests in situations where the set of bidders is unknown. Nevertheless, one may ask how our results would be affected if we adopt the second-price payment rule.

In the case of formal auctions nothing changes. In Chi, Murto, and Välimäki (2019), we show formally that in the case of two bidder types, the revenue properties in the bidding equilibria of the first-price auction and second-price auction are equivalent. This implies that also with endogenous entry, the set of equilibrium is the same in first-price and second-price formal auctions. We should note that this equivalence relies on our finding that only two type of bidders are present in the bidding stage, as established in Lemma 5. We know from Milgrom and Weber (1982) that such equivalence would not hold if the participating bidders draw their types from a more general signal space.

In our previous working paper, we also analyzed the informal second-price auction. The main difficulty for that analysis arises from the multiplicity of symmetric bidding equilibria. In contrast to informal first-price auctions, this multiplicity cannot be ruled out even in the case of interim entry decisions. However, it is clear that no matter how we select the equilibrium, the high type bidders always get a positive information rent on top of their social contribution, and hence entry rates are distorted. This means that the informal first-price auction dominates the informal second-price auction in expected revenue whenever the former has no distortions.

We can further show that the informal second-price auction has a bidder optimal equilibrium, i.e. there is a single Pareto efficient equilibrium for the bidders. If we select this equilibrium for the bidding stage, then we get unambiguous revenue ranking results between

the two informal auction formats: the informal first-price auction raises a higher revenue than the informal second price auction.

7.3   Asymmetric Equilibria and Participation by Invitation

In models with entry costs, one often finds that asymmetric equilibria profiles dominate symmetric ones in terms of social efficiency. We are motivated by models where the set of potential participants is dispersed and it is not possible to achieve the ex ante co-ordination required for asymmetric equilibria.

One possible remedy could then be that the seller approaches only a subset of potential bidders and restricts participation to them only. Since the seller does not know the private information of the potential bidders, it is not clear if such a strategy would raise the expected revenues. Moreover, if only a small fraction of all potentially interested bidders are truly interested and if the seller incurs a cost of contacting the bidders, it is clearly better to rely on self-selecting mechanisms of the type that we analyze in this paper.

Appendix: Omitted Proofs

Proof of Lemma 2. If $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ is an equilibrium of the reduced model (i.e. the model with $\theta \in \{\theta_1, \theta_M\}$), then types $\theta = 1$ and $\theta = M$ get the same payoff in the full model under strategy profile $\left(\bar{b}, (\bar{\pi}_1, 0, ..., 0, \bar{\pi}_M)\right)$ and hence their behavior remains consistent with equilibrium. We have to prove that no other type $\theta_m \in \{\theta_2, ..., \theta_{M-1}\}$ has a strict incentive to enter against $\left(\bar{b}, (\bar{\pi}_1, 0, ..., 0, \bar{\pi}_M)\right)$. Denote by $\overline{R}_0(p)$ and $\overline{R}_1(p)$ the expected payoff of a bidder who bids p against $\left(\bar{b}, (\bar{\pi}_1, 0, ..., 0, \bar{\pi}_M)\right)$, conditional on state $\omega = 0$ and $\omega = 1$, respectively. Suppose that there is a type $\theta_m \in \{\theta_2, ..., \theta_{M-1}\}$, and a bid p_m such that θ_m gets strictly more than c when the others follow $\left(\bar{b}, (\bar{\pi}_1, 0, ..., 0, \bar{\pi}_M)\right)$. This means that $q_m \overline{R}_1(p_m) + (1 - q_m)\overline{R}_0(p_m) > c$. But since $q_0 < q_m < q_M$, then either $q_1 \overline{R}_1(p_m) + (1 - q_0)\overline{R}_0(p_m) > c$ or $q_M \overline{R}_1(p_m) + (1 - q_M)\overline{R}_0(p_m) > c$, implying that either type θ_1 or θ_M would get a payoff strictly greater than c by following bidding strategy p_m. This contradicts the fact that $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ is an equilibrium of the reduced model.

To prove the second part of the proposition, suppose that there is an equilibrium (b, π) in the full model such that $\pi_m > 0$ for some $m \in \{2, ..., M - 1\}$. Let $R_\omega(p)$ denote the expected payoff of bidding p conditional on state ω:

$$R_1(p) = e^{-\sum_{m=1}^{M} \alpha_{1,m} \pi_m (1 - F_m(p))} (v(1) - p),$$

$$R_0(p) = e^{-\sum_{m=1}^{M} \alpha_{0,m} \pi_m (1 - F_m(p))} (v(0) - p).$$

Since (b, π) is an equilibrium, we must have for every $p \geq 0$,

$$q_m R_1(p) + (1 - q_m) R_0(p) \leq c, \text{ for } m = 1, ..., M$$

and for any $p \in \mathrm{supp} F_m$, we have

$$q_m R_1(p) + (1 - q_m) R_0(p) = c.$$

These equations imply that if $p \in \mathrm{supp} F_m$, $m \in \{2, ..., M - 1\}$, then

$$R_1(p) = R_0(p) = c,$$

and hence any type will be indifferent between entering and bidding p and staying out. It follows that if there is a strategy profile $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ in the reduced model that induces the same state-by-state payoffs $R_1(p)$ and $R_0(p)$ than (b, π), then $\left(\bar{b}, (\bar{\pi}_1, \bar{\pi}_M)\right)$ is an equilibrium of the reduced model. It remains to show that we can construct such a strategy profile.

Since $\frac{\alpha_{1,1}}{\alpha_{0,1}} < ... < \frac{\alpha_{1,M}}{\alpha_{0,M}}$, we can always choose $\widehat{\pi}_1 > 0$ and $\widehat{\pi}_M > 0$ such that

$$\alpha_{1,1}\widehat{\pi}_1 + \alpha_{1,M}\widehat{\pi}_M = \sum_{m=1}^{M} \alpha_{1,m}\pi_m,$$

$$\alpha_{0,1}\widehat{\pi}_1 + \alpha_{0,M}\widehat{\pi}_M = \sum_{m=1}^{M} \alpha_{0,m}\pi_m.$$

Similarly, we choose c.d.f.:s $\widehat{F}_1(p)$ and $\widehat{F}_M(p)$ such that for all $p \geq 0$, we have

$$\alpha_{1,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) + \alpha_{1,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right) = \sum_{m=1}^{M} \alpha_{1,m}\pi_m\left(1 - F_m(p)\right),$$

$$\alpha_{0,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) + \alpha_{0,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right) = \sum_{m=1}^{M} \alpha_{0,m}\pi_m\left(1 - F_m(p)\right).$$

We now have a strategy profile of the reduced model, $\left(\left\{\widehat{F}_1(\cdot), \widehat{F}_M(\cdot)\right\}, \{\widehat{\pi}_1, \widehat{\pi}_M\}\right)$, with the same state-by-state payoffs as the original equilibrium of the full model:

$$\widehat{R}_1(p) \quad : \quad = e^{-\alpha_{1,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) - \alpha_{1,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right)}(v(1) - p) = R_1(p),$$

$$\widehat{R}_0(p) \quad : \quad = e^{-\alpha_{0,1}\widehat{\pi}_1\left(1 - \widehat{F}_1(p)\right) - \alpha_{0,M}\widehat{\pi}_M\left(1 - \widehat{F}_M(p)\right)}(v(0) - p) = R_0(p).$$

∎

Proof of Lemma 3. Let F_l and F_h denote the bid distributions of an atomless bidding equilibrium b corresponding to entry profile $\pi = (\pi_h, \pi_l)$ and let $\lambda_{\omega,\theta} = \alpha_{\omega,\theta}\pi_\theta$, $\omega \in \{0, 1\}$, $\theta \in \{h, l\}$. We start by ruling out the possibility of gaps in the union of the bid supports. This is by a standard argument: suppose that there is some interval (p, p'') such that no type bids within that interval, but p'' is in the bid support for type θ. Then type θ could reduce her bid from p'' without decreasing her winning probability and this would strictly increase her expected payoff. It follows that

$$suppF_l \cup suppF_h = [0, p^{\max}] \text{ for some } p^{\max} > 0.$$

Note that this does not rule out gaps in the bid supports of individual types.

We next investigate the possible configurations of the bid supports $suppF_l$ and $suppF_h$. We first specify the range of bids where the supports may overlap, i.e. where both types can potentially be indifferent simultaneously. Denote by $U_\theta(p)$ the payoff of type θ who bids p, when bidding distributions are given by $F_\theta(p)$:

$$\begin{aligned} U_\theta(p) = \quad & q_\theta e^{-\lambda_{1,h}(1 - F_h(p)) - \lambda_{1,l}(1 - F_l(p))}(v(1) - p) \\ & + (1 - q_\theta) e^{-\lambda_{0,h}(1 - F_h(p)) - \lambda_{0,l}(1 - F_l(p))}(v(0) - p). \end{aligned} \quad (4)$$

Differentiating with respect to p, we have:

$$U'_\theta(p) = q_\theta e^{-\lambda_{1,h}(1-F_h(p))-\lambda_{1,l}(1-F_l(p))} \left[(F'_h(p)\lambda_{1,h} + F'_l(p)\lambda_{1,l})(v(1)-p) - 1\right]$$

$$+ (1-q_\theta) e^{-\lambda_{0,h}(1-F_h(p))-\lambda_{0,l}(1-F_l(p))} \left[(F'_h(p)\lambda_{0,h} + F'_l(p)\lambda_{0,l})(v(0)-p) - 1\right] \quad (5)$$

In equilibrium, $F'_\theta(p) > 0$ requires $U'_\theta(p) = 0$ to maintain indifference within bidding support. To analyze when this can hold, we denote the two terms in square brackets by $B(1)$ and $B(0)$:

$$B(1) = (F'_h(p)\lambda_{1,h} + F'_l(p)\lambda_{1,l})(v(1)-p) - 1,$$
$$B(0) = (F'_h(p)\lambda_{0,h} + F'_l(p)\lambda_{0,l})(v(0)-p) - 1.$$

These terms are weighted in (5) by positive terms that depend on θ only through q_θ. Since $q_h > q_l$, we note that $U'_h(p)$ puts more weight on term $B(1)$ than $B(0)$, relative to $U'_l(p)$. It is immediate that for $U'_h(p)$ and $U'_l(p)$ to be zero simultaneously, it must be that $B(1) = B(0) = 0$. Since $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$, this is possible only if

$$\lambda_{0,l}(v(0)-p) > \lambda_{1,l}(v(0)-p).$$

This can hold only for low values of p. Let us define $\widetilde{p}$ as the cutoff value such that the above inequality holds for $p < \widetilde{p}$:

$$\widetilde{p} = \max\left(0, \frac{v(0)\lambda_{0,l} - v(1)\lambda_{1,l}}{\lambda_{0,l} - \lambda_{1,l}}\right).$$

(Note that we define $\widetilde{p} = 0$ if indifference is never possible). We summarize the implications of this reasoning in the following Claim. Part 1 says that the overlap of bidding supports is possible only for $p < \widetilde{p}$. Part 2 says that in a range where only low type randomizes, value of high type is U-shaped with minimum at $p = \widetilde{p}$. Part 3 says that in a range where only high type randomizes, value of low type is decreasing and hence the low type prefers lower bids.

Claim 1 *Let $\lambda_{1,h} > \lambda_{0,h}$ and $\lambda_{0,l} > \lambda_{1,l}$ be given, and let $F_\theta(p)$, $\theta = h, l$, be an atomless equilibrium bidding distribution. We have:*

1. *If $F'_\theta(p) > 0$ for $\theta = h, l$, then $p < \widetilde{p}$.*

2. *If $F'_l(p) > 0$ and $F'_h(p) = 0$, then*

$$U'_h(p) \begin{cases} < 0 \text{ for } p < \widetilde{p} \\ > 0 \text{ for } p > \widetilde{p} \end{cases}.$$

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3. If $F_h'(p) > 0$ and $F_l'(p) = 0$, then $U_l'(p) < 0$.

Proof. Part 1: $F_\theta'(p) > 0$ for $\theta = h, l$ requires that $B(1) = B(0) = 0$, which is only possible if $p < \widetilde{p}$. Part 2: If $F_l'(p) > 0$, then $U_l'(p) = 0$. If $F_h'(p) = 0$, then $U_l'(p) = 0$ implies that $B(1) < 0 < B(0)$ for $p < \widetilde{p}$, and $B(0) < 0 < B(1)$ for $p > \widetilde{p}$. Since $q_h > q_l$, the result follows. Part 3: If $F_h'(p) > 0$, then $U_h'(p) = 0$. If $F_l'(p) = 0$, then $U_h'(p) = 0$ implies that $B(0) < 0 < B(1)$. Since $q_l < q_h$, the result follows. ∎

With this preliminary result in place, we can prove the rest of the Lemma. Part 3 of Claim 1 immediately confirms that $suppF_l = [0, p']$ for some $p' > 0$ (for the rest of the proof, we continue to use p' to denote the upper bound of the low type bid support). To see this, suppose to the contrary that there is some non-empty interval (p_1, p_2) such that $(p_1, p_2) \cap suppF_l = \emptyset$ while $p_2 \in suppF_l$. Since the union of bid supports is an interval, we then have $F_h'(p) > 0$ for $p \in (p_1, p_2)$, and by part 3 of Claim 1 we have $U_l'(p) < 0$ for $p \in (p_1, p_2)$. But then the low type has a profitable deviation from p_2 to p_1 and we have a contradiction.

We will now show that if

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} < \frac{v(1)}{v(0)}, \tag{6}$$

then the bid supports of the two types cannot overlap. Suppose to the contrary that the supports do overlap. By part 1 of Claim 1, overlap is possible only for $p < \widetilde{p}$, and combining this with part 2 we must have $0 \in suppF_h$ whenever the supports overlap. As before, we denote by p' the upper bound of the low type bid support, i.e. $suppF_l = [0, p']$. Since there are no gaps in the union of the bid supports, we also have $p' \in suppF_h$. Both types must then be indifferent between bidding 0 and p', which gives the following indifference condition:

$$q_\theta e^{-\lambda_{1,h} - \lambda_{1,l}} v(1) + (1 - q_\theta) e^{-\lambda_{0,h} - \lambda_{0,l}} v(0)$$
$$= q_\theta e^{-(1 - F_h(p'))\lambda_{1,h}} (v(1) - p') + (1 - q_\theta) e^{-(1 - F_h(p'))\lambda_{0,h}} (v(0) - p')$$

for $\theta = l, h$. This can be rewritten as

$$0 = q_\theta e^{-\lambda_{1,h}} \left[\left(e^{F_h(p')\lambda_{1,h}} - e^{-\lambda_{1,l}} \right) v(1) - e^{F_h(p')\lambda_{1,h}} p' \right]$$
$$+ (1 - q_\theta) e^{-\lambda_{0,h}} \left[\left(e^{F_h(p')\lambda_{0,h}} - e^{-\lambda_{0,l}} \right) v(0) - e^{F_h(p')\lambda_{0,h}} p' \right]. \tag{7}$$

For this to hold simultaneously for both types $\theta = l, h$, both of the two terms in square-brackets must be zero. Suppose that the second term is zero, that is

$$\left[\left(e^{F_h(p')\lambda_{0,h}} - e^{-\lambda_{0,l}} \right) v(0) - e^{F_h(p')\lambda_{0,h}} p' \right] = 0.$$

Multiplying by $e^{-F_h(p')\lambda_{0,h}}$, this is equivalent to

$$\left[\left(1 - e^{-\lambda_{0,l}}e^{-F_h(p')\lambda_{0,h}}\right)v(0) - p'\right] = 0.$$

Combining this with (6) and using the fact that $e^{-F_h(p')\lambda_{1,h}} < e^{-F_h(p')\lambda_{0,h}} < 1$, this implies that

$$\left[\left(1 - e^{-\lambda_{1,l}}e^{-F_h(p')\lambda_{1,h}}\right)v(1) - p'\right] > 0,$$

and so multiplying by $e^{F_h(p')\lambda_{1,h}}$ we see that the first term in square-brackets in (7) is strictly positive:

$$\left[\left(e^{F_h(p')\lambda_{1,h}} - e^{-\lambda_{1,l}}\right)v(1) - e^{F_h(p')\lambda_{1,h}}p'\right] > 0.$$

This is a contradiction. We can conclude that it is not possible to make both types indifferent between bidding 0 and p', and so the bid supports must be non-overlapping intervals with a single point in common.

We will next show that if

$$\frac{1 - e^{-\lambda_{0,l}}}{1 - e^{-\lambda_{1,l}}} > \frac{v(1)}{v(0)}, \tag{8}$$

then the bid supports must overlap. Suppose, to the contrary, that there is an equilibrium, where the bid-supports are non-overlapping intervals: $suppF_l = [0, p']$ and $suppF_h = [p', p^{\max}]$ for some $0 < p' < p^{\max}$. The low type must be indifferent between bidding 0 and p':

$$q_l e^{-\lambda_{1,h} - \lambda_{1,l}} v(1) + (1 - q_l) e^{-\lambda_{0,h} - \lambda_{0,l}} v(0)$$
$$= q_l e^{-\lambda_{1,h}} (v(1) - p') + (1 - q_l) e^{-\lambda_{0,h}} (v(0) - p'),$$

which can be rewritten as

$$q_l e^{-\lambda_{1,h}} \left[\left(1 - e^{-\lambda_{1,l}}\right)v(1) - p'\right] + (1 - q_l) e^{-\lambda_{0,h}} \left[\left(1 - e^{-\lambda_{0,l}}\right)v(0) - p'\right] = 0.$$

For this to hold, one of the terms in square-brackets must be positive and the other one must be negative. Combining this with (8), we must have

$$\left(1 - e^{-\lambda_{0,l}}\right)v(0) - p' > 0 > \left(1 - e^{-\lambda_{1,l}}\right)v(1) - p'.$$

But then, since $q_h > q_l$, this implies that

$$q_h e^{-\lambda_{1,h}} \left[\left(1 - e^{-\lambda_{1,l}}\right)v(1) - p'\right] + (1 - q_h) e^{-\lambda_{0,h}} \left[\left(1 - e^{-\lambda_{0,l}}\right)v(0) - p'\right] < 0$$

or

$$q_h e^{-\lambda_{1,h} - \lambda_{1,l}} v(1) + (1 - q_h) e^{-\lambda_{0,h} - \lambda_{0,l}} v(0)$$
$$> q_h e^{-\lambda_{1,h}} (v(1) - p') + (1 - q_h) e^{-\lambda_{0,h}} (v(0) - p'),$$

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and so the high type has a strictly profitable deviation to bidding 0 rather than p'. We can conclude that an equilibrium with non-overlapping intervals is not possible. The only possibility is an equilibrium with overlapping bid supports, and as we have already concluded above, in such a case 0 must be contained in the supports of both types. ∎

Proof of Lemma 4. If $\pi_l^*(\pi_h) = 0$, it is easy to check that (2) holds since $W(\pi_h, 0)$ is concave in π_h. If $\pi_l^*(\pi_h) > 0$, the first order condition for π_l must hold:

$$
\begin{aligned}
0 &= \alpha_{1,l} q \left(v(1) e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l^*(\pi_h)} - c \right) \\
&\quad + \alpha_{0,l} (1-q) \left(v(0) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l^*(\pi_h)} - c \right).
\end{aligned}
\tag{9}
$$

If $\pi_h = \pi_h^{opt}$, then $\pi_l^*(\pi_h) = \pi_l^{opt}$ and (9) is satisfied since

$$
v(1) e^{-\alpha_{1,h}\pi_h^{opt} - \alpha_{1,l}\pi_l^{opt}} - c = v(0) e^{-\alpha_{0,h}\pi_h^{opt} - \alpha_{0,l}\pi_l^{opt}} - c = 0.
$$

If $\pi_h > (<) \pi_h^{opt}$, we note that since $\alpha_{1,h} > \alpha_{0,h}$, (9) can only hold if

$$
v(1) e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l^*(\pi_h)} - c < (>) 0 < (>) v(0) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l^*(\pi_h)} - c.
$$

But then, since $\frac{\alpha_{1,h}}{\alpha_{0,h}} > \frac{\alpha_{1,l}}{\alpha_{0,l}}$, comparison with (9) implies

$$
\begin{aligned}
\frac{\partial W(\pi_h, \pi_l^*(\pi_h))}{\partial \pi_h} &= \alpha_{1,h} q \left(v(1) e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\pi_l^*(\pi_h)} - c \right) \\
&\quad + \alpha_{0,h} (1-q) \left(v(0) e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\pi_l^*(\pi_h)} - c \right) < (>) 0.
\end{aligned}
$$

The result then follows from the envelope theorem. ∎

Proof of Proposition 1. In any bidding equilibrium, the equilibrium payoff to the low types is $V_l^{IF}(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l)$. Therefore, if an equilibrium (b, π) exists, then the entry point $\pi = \left(\pi_h^{IF}, \pi_l^{IF}\right)$ must be located along the planner's reaction curve $\pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right)$. Moreover, in the part of that curve, where $\pi_h < \pi_h^{opt}$, we have $V_h^0(\pi_h, \pi_l^*(\pi_h)) > 0$, and so the high type would get strictly more than c by bidding zero. We can conclude that in any equilibrium the entry profile must satisfy

$$
\pi_h^{IF} \geq \pi_h^{opt}, \ \pi_l^{IF} = \pi_l^*\left(\pi_h^{IF}\right).
\tag{10}
$$

We now establish the existence and uniqueness of an equilibrium. We do that separately in two different cases corresponding to the two cases presented in Lemma 3.

Let us initially fix the entry profile to the socially optimal one, and assume first that

$$
\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)} \text{ for } \pi_l = \pi_l^{opt}.
\tag{11}
$$

Since the left-hand side of (11) is decreasing in π_l, this implies that (11) holds for all entry profiles satisfying (10) and hence by Lemma 3 zero must be included in the bid supports for both types in any candidate for a bidding equilibrium. But if zero is in the support of the high type bidder, the high type bidder will get strictly less than c in any profile $\pi_h > \pi_h^{opt}$, $\pi_l = \pi_l^*(\pi_h)$. The only possibility is that in equilibrium entry profile is socially optimal and zero is in the bid supports of both type of bidders. We now establish existence and uniqueness of such an equilibrium by constructing one. Since the construction is unique, this also establishes uniqueness of equilibrium under (11).

Assume that entry rates are socially optimal i.e. given by

$$\lambda_{\omega,\theta} = \alpha_{\omega,\theta}\pi_\theta^{opt}, \ \omega = 0,1, \theta = l,h,$$

and (11) holds for $\pi_l = \pi_l^{opt}$. Given these entry rates, define p^{max} such that a high type bidder just breaks even by bidding p^{max} if she wins for sure:

$$p^{max} = q_h v(1) + (1 - q_h) v(0) - c.$$

Clearly, p^{max} must be the upper bound of the high type bid support in equilibrium. We now construct an equilibrium by proceeding downwards from p^{max}, assuming that only the high type is active near p^{max} and defining $F_h(p)$ in such a way that the high types get c for all such p:

$$
\begin{aligned}
U_h(p) &= q_h e^{-\lambda_{1,h}(1-F_h(p))}(v(1) - p) \\
&\quad + (1 - q_h) e^{-\lambda_{0,h}(1-F_h(p))}(v(0) - p) \\
&= c \text{ for } p < p^{max}.
\end{aligned}
$$

Note that we have $U_l(p^{max}) < c$ so that the low type has no incentive to bid near p^{max}. By part 3 of Claim 1 (in the proof of Lemma 3), we have $U_l'(p) < 0$ over this range, and as we move p downwards from p^{max} we will reach a point $\overrightarrow{p}$ such that $U_l(\overrightarrow{p}) = c$. This point is pinned down by the condition that low type is indifferent between bidding $\overrightarrow{p}$ and zero:

$$q_l e^{-\lambda_{1,h}(1-F_h(\overrightarrow{p}))}(v(1) - \overrightarrow{p}) + (1 - q_l) e^{-\lambda_{0,h}(1-F_h(\overrightarrow{p}))}(v(0) - \overrightarrow{p}) = c.$$

We have then two different cases depending on whether $\overrightarrow{p}$ is below or above $\widetilde{p}$.

Case 1: $\overrightarrow{p} \leq \widetilde{p}$. We can define $F_l(p)$ and $F_h(p)$ below $\overrightarrow{p}$ so that both types are indifferent for all $p \leq \overrightarrow{p}$, that is,

$$
\begin{aligned}
U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))}e^{-\lambda_{1,l}(1-F_l(p))}(v(1) - p) \\
&\quad + (1 - q_\theta) e^{-\lambda_{0,h}(1-F_h(p))}e^{-\lambda_{0,l}(1-F_l(p))}(v(0) - p) \\
&= c \text{ for } 0 \leq p \leq \overrightarrow{p}, \theta = l,h.
\end{aligned}
$$

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Since entry is socially optimal, the above equations automatically hold for $p = 0$, $F_l(0) = F_h(0) = 0$. As a result, we end up with an equilibrium, where the low-type bid support is $[0, \overrightarrow{p}]$ and high-type bid support is $[0, p^{\max}]$.

Case 2: $\overrightarrow{p} > \widetilde{p}$. By part 1 of Claim 1, we cannot have indifference simultaneously for both types above $\widetilde{p}$, and hence the same structure as in Case 1 is not possible. Instead, we will construct an interval $[\overleftarrow{p}, \overrightarrow{p}]$ containing $\widetilde{p}$, where only the low type is active: define $F_l(p)$ within $[\overleftarrow{p}, \overrightarrow{p}]$ such that

$$
\begin{aligned}
U_l(p) &= q_l e^{-\lambda_{1,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{1,l}(1-F_l(p))}\left(v(1)-p\right) \\
&\quad + (1-q_h)\, e^{-\lambda_{0,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{0,l}(1-F_l(p))}\left(v(0)-p\right) \\
&= c \text{ for } p \in \left[\overleftarrow{p}, \overrightarrow{p}\right].
\end{aligned}
$$

By part 2 of Claim 1, $U_h'(p) > 0$ for $p > \widetilde{p}$ and $U_h'(p) < 0$ for $p < \widetilde{p}$. We then define $\overleftarrow{p}$ as the point where

$$
\begin{aligned}
U_h\left(\overleftarrow{p}\right) &= q_l e^{-\lambda_{1,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{1,l}\left(1-F_l\left(\overleftarrow{p}\right)\right)}\left(v(1)-\overleftarrow{p}\right) \\
&\quad + (1-q_h)\, e^{-\lambda_{0,h}\left(1-F_h\left(\overrightarrow{p}\right)\right)-\lambda_{0,l}\left(1-F_l\left(\overleftarrow{p}\right)\right)}\left(v(0)-\overleftarrow{p}\right) \\
&= c.
\end{aligned}
$$

Since $\overleftarrow{p} < \widetilde{p}$, we can define $F_l(p)$ and $F_h(p)$ below $\overleftarrow{p}$ so that both types are indifferent, that is

$$
\begin{aligned}
U_\theta(p) &= q_\theta e^{-\lambda_{1,h}(1-F_h(p))} e^{-\lambda_{1,l}(1-F_l(p))}\left(v(1)-p\right) \\
&\quad + (1-q_\theta)\, e^{-\lambda_{0,h}(1-F_h(p))} e^{-\lambda_{0,l}(1-F_l(p))}\left(v(0)-p\right) \\
&= c \text{ for } 0 \leq p \leq \overleftarrow{p},\ \theta = l, h.
\end{aligned}
$$

As a result we have an equilibrium, where the low type bidding support is $[0, \overrightarrow{p}]$ and high type bidding support is disconnected and given by $[0, \overleftarrow{p}] \cup [\overrightarrow{p}, p^{\max}]$.

Let us next consider the case, where

$$
\frac{1-e^{-\alpha_{0,l}\pi_l}}{1-e^{-\alpha_{1,l}\pi_l}} < \frac{v(1)}{v(0)}, \text{ for } \pi_l = \pi_l^{opt}. \tag{12}
$$

Then, by Lemma 3, a bidding equilibrium with zero in the support of the high type is not possible under the socially optimal entry profile, and consequently a high type bidder would get strictly more than c under the socially optimal entry profile. A bidding equilibrium with zero in the high type support is neither possible for an entry profile $\pi_h > \pi_h^{opt}$, $\pi_l = \pi_l^*(\pi_h)$, since the high type would get strictly less than c in such a case. The only possibility is that

$\pi_h > \pi_h^{opt}$, $\pi_l = \pi_l^*(\pi_h)$, and the bid supports are non-overlapping intervals with a single point in common, i.e. $suppF_l = [0, p']$ and $suppF_h = [p', p^{\max}]$ for some $0 < p' < p^{\max}$. We will now construct such an equilibrium.

Since p' is contained in the supports of both types, both types must obtain expected payoff equal to c by bidding p'. This means that expected payoff conditional on state must be equal to c in both states (otherwise the type that has a higher posterior on the state that gives a higher payoff would get strictly more than the other type). Therefore, we have:

$$\begin{aligned}
e^{-\alpha_{1,h}\pi_h}\left(v\left(1\right) - p'\right) &= c, \\
e^{-\alpha_{0,h}\pi_h}\left(v\left(0\right) - p'\right) &= c.
\end{aligned} \tag{13}$$

We now show that there is a unique pair of values p' and π_h that satisfies this pair of equations. Eliminating p' gives us:

$$e^{\alpha_{1,h}\pi_h} - e^{\alpha_{0,h}\pi_h} = \frac{v\left(1\right) - v\left(0\right)}{c}. \tag{14}$$

Since $\alpha_{1,h} > \alpha_{0,h}$, the left hand side is strictly increasing in π_h, and so there clearly exists a unique solution $\pi_h > 0$. Given this, we can solve p' from

$$p' = v\left(0\right) - e^{\alpha_{0,h}\pi_h}c.$$

Note that as long as $c < \bar{c}$, this solution always satisfies $0 < p' < v\left(0\right)$ (when $c = \bar{c}$, we have $p' = 0$). Our candidate for equilibrium entry profile is then $\left(\pi_h^{IF}, \pi_l^{IF}\right) = \left(\pi_h, \pi_l^*\left(\pi_h\right)\right)$, where π_h solves (14). It is easy to check that as long as $c < \bar{c}$, we have $\pi_l^*\left(\pi_h\right) > 0$.

It is now straightforward to finish the construction of the bidding equilibrium. For $0 \le p \le p'$, define $F_l\left(p\right)$ so that

$$\begin{aligned}
U_l\left(p\right) &= q_l e^{-\alpha_{1,h}\pi_h - \alpha_{1,l}\left(1 - F_l\left(p\right)\right)\pi_l}\left(v\left(1\right) - p\right) \\
&\quad + \left(1 - q_l\right)e^{-\alpha_{0,h}\pi_h - \alpha_{0,l}\left(1 - F_l\left(p\right)\right)\pi_l}\left(v\left(0\right) - p\right). \\
&= c.
\end{aligned}$$

Since $p' < v\left(0\right)$, there is a unique increasing $F_l\left(p\right)$ that satisfies this equation for $0 \le p \le p'$. Note that since $\pi_l = \pi_l^*\left(\pi_h\right)$, we have $F_l\left(0\right) = 0$ and since π_h satisfies (13), we have $F_l\left(p'\right) = 1$. Similarly, for $p' < p \le p^{\max}$, define $F_h\left(p\right)$ so that

$$\begin{aligned}
U_h\left(p\right) &= q_h e^{-\alpha_{1,h}\pi_h\left(1 - F_h\left(p\right)\right)}\left(v\left(1\right) - p\right) \\
&\quad + \left(1 - q_h\right)e^{-\alpha_{0,h}\pi_h\left(1 - F_h\left(p\right)\right)}\left(v\left(0\right) - p\right) \\
&= c,
\end{aligned}$$

where $p^{\max} = q_h v(1) + (1 - q_h) v(0) - c$. Again, we can check that there is a unique increasing function $F_h(p)$ that satisfies this equation for $p' \leq p \leq p^{\max}$, with $F_h(p') = 0$ and $F_h(p^{\max}) = 1$. By assumption (12) and part 2 of Claim 1, the high type has no profitable deviation to $[0, p']$, and by part 3 of Claim 1 the low type has no profitable deviation to $[p', p^{\max}]$. Obviously no type has a profitable deviation above $p^{\max}$ and hence this is an equilibrium. Since this construction is unique, the model has a unique symmetric equilibrium. $\blacksquare$

Proof of Lemma 5. Suppose that there is an equilibrium with $\pi_m > 0$ for some $m \in \{2, ..., M - 1\}$. Let V_ω, $\omega = 0, 1$, denote the ex-ante equilibrium payoff conditional on state for bidding according to type θ_m strategy. Since θ_m breaks just even in equilibrium, $q_m V_1 + (1 - q_m) V_0 = c$. We have $q_1 < q_m < q_M$. Hence, if $V_1 > V_0$, then type θ_M can get strictly more than c by mimicking the bidding strategy of type θ_m, and if $V_0 > V_1$, then type θ_1 can get strictly more than c by mimicking the bidding strategy of type θ_m. Therefore, it must be that $V_0 = V_1$. This of course means that any type can get c by mimicking type θ_m bidding strategy. It remains to show that some type gets strictly more by deviating from type θ_m bidding strategy.

Note that since we consider formal auctions, the bidding strategy is conditional on n, the realized number of entrants. Let $R_\omega^n(p)$ denote the expected payoff of bidding p, conditional on n and conditional on ω. Suppose that there is some n and either some interval (p', p'') or an atom p''' in the support of type θ_m bidding strategy such that $R_1^n(p) \neq R_0^n(p)$ within (p', p'') or at p'''. Then, either type θ_1 or type θ_M gets in expectation more than θ_m by deviating to (p', p'') or p''' for this n. By using this deviation for n, and by mimicking θ_m for all other n, this type (either θ_1 or θ_M) gets a strictly higher ex-ante payoff than θ_m, contradicting the assumption that this strategy is an equilibrium in the overall game with interim entry. It follows that unless $R_1^n(p) = R_0^n(p)$ within the whole support of type θ_m for all n, there cannot exist an equilibrium with $\pi_m > 0$, $m \in \{2, ..., M - 1\}$.

It remains to show that we cannot have $R_1^n(p) = R_0^n(p)$ within the whole support of type θ_m for all n. First, note that for any $n \geq 2$, $\underline{p}_n := \min \bigcup_{i=1}^{M} \mathrm{supp} F_i^n > v(0)$, where F_i^n is the bidding distribution of type i for realized number of entrants n. If this were not the case, then anyone bidding $\underline{p}_n$ would have a strictly profitable deviation up (recall that $n \geq 2$, so if there is no atom at $\underline{p}_n$, then one can never win by bidding there, and if there is an atom, a small deviation increases probability of winning discretely). But for all $p > v(0)$, we always have $R_0^n(p) < 0$, and hence $R_1^n(p) \neq R_0^n(p)$. $\blacksquare$

Proof of Proposition 2. Assume that only the highest and lowest types, h and l, enter with a strictly positive rate. Let $V_\theta^F(\pi_h, \pi_l)$ denote the expected payoff of type θ in the

formal auction with entry profile $\pi = (\pi_h, \pi_l)$. We proceed to show that there is an entry profile $\left(\pi_h^F, \pi_l^F\right)$ satisfying $V_h^F\left(\pi_h^F, \pi_l^F\right) = V_l^F\left(\pi_h^F, \pi_l^F\right) = c$. It is clear that in that case an intermediate type $\theta_m \in \{\theta_2, ..., \theta_{M-1}\}$ could not get more than c by entering. To see this, suppose to the contrary that $V_h^F\left(\pi_h^F, \pi_l^F\right) = V_l^F\left(\pi_h^F, \pi_l^F\right) = c$, but type θ_m gets strictly more than c using bidding strategy b_m against (π_h, π_l). Let V_ω denote the ex-ante payoff conditional on state ω using bidding strategy b_m. If $V_1 \geq V_0$, then type h would get strictly more than c using b_m, a contradiction. Similarly, if $V_0 \geq V_1$, then type l would get strictly more than c using b_m, a contradiction.

The unique bidding equilibrium of a formal auction is characterized in Lemma 6, and it follows from that Lemma that the low type gets payoff equal to zero on expectation whenever $n \geq 2$ whereas the high type gets strictly more than zero for any $n \geq 2$. If $n = 1$, both types get their social contribution to total welfare. Hence, for a given entry profile $\pi = (\pi_h, \pi_l)$, the low type gets $V_l^F(\pi_h, \pi_l) = V_l^0(\pi_h, \pi_l)$ and the high type gets $V_l^F(\pi_h, \pi_l) > V_l^0(\pi_h, \pi_l)$. This implies that an equilibrium entry profile must satisfy

$$\pi_h^F > \pi_h^{opt}, \ \ \pi_l^F = \pi_l^*\left(\pi_h^{IF}\right). \tag{15}$$

It remains to show that there is a π_h^F such that $V_h^F\left(\pi_h^F, \pi_l^*\left(\pi_h^{IF}\right)\right) = c$. Note from Lemma 6 that the lower bound of the high type bidding support, $\underline{p}(n)$, as well as the probability of winning with such a bid, are continuous in (π_h, π_l). Therefore, $V_h^F\left(\pi_h, \pi_l^*\left(\pi_h\right)\right)$ is continuous in π_h. It remains to show that $V_h^F\left(\pi_h, \pi_l^*\left(\pi_h\right)\right)$ crosses c as we vary π_h. As already shown, at the efficient entry profile $\left(\pi_h^{opt}, \pi_l^*\left(\pi_h^{opt}\right)\right)$, we have $V_h^F\left(\pi_h^{opt}, \pi_l^*\left(\pi_h^{opt}\right)\right) > c$. On the other hand, when π_h is high enough, we must have $V_h^F\left(\pi_h, \pi_l^*\left(\pi_h\right)\right) < c$. To see this, note first that if only high types enter, the payoff of the high type is equal to her social contribution:

$$V^F\left(\pi_h, 0\right) = V_h^0\left(\pi_h, 0\right) \text{ for all } \pi_h > 0.$$

Denote by $\overline{\pi}_h$ the critical entry rate of the high type such that low types do not want to enter at all:

$$\overline{\pi}_h := \inf\left\{\pi_h : \pi_l^*\left(\pi_h\right) = 0\right\}.$$

Therefore, we have

$$\lim_{\pi_h \uparrow \overline{\pi}_h} V_h^F\left(\pi_h, \pi_l^*\left(\pi_h\right)\right) = V_h^0\left(\overline{\pi}_h, 0\right).$$

Under our assumption $c < \overline{c}$, we have $\pi_h^*(0) < \overline{\pi}_h$. Since $V_h^0\left(\pi_h, \pi_l\right)$ is strictly decreasing in π_h, it follows that $V_h^0\left(\overline{\pi}_h, 0\right) < c$, and so $V_h^0\left(\pi_h, \pi_l^*\left(\pi_h\right)\right) < c$ for π_h sufficiently high.

The result now follows from the intermediate value theorem: there must be some $\pi_h^F \in \left(\pi_h^{opt}, \overline{\pi}_h\right)$ such that $V_h^F\left(\pi_h^F, \pi_l^*\left(\pi_h^F\right)\right) = c$. Setting $\pi_l^F = \pi_l^*\left(\pi_h^F\right)$, we have $V_h^F\left(\pi_h^F, \pi_l^F\right) =$

$V_l^F \left(\pi_h^F, \pi_l^F \right) = c$. Coupled with the bidding equilibrium characterized in Lemma 6, this is an equilibrium. Since the reaction curve $\pi_l^* (\pi_h)$ is strictly decreasing in π_h, the entry rates satisfy $\pi_h > \pi_h^{opt}$, $\pi_l^H < \pi_l^{opt}$. ∎

Proof of Proposition 3. Note that

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}}$$

is decreasing in π_l with $\lim_{\pi_l \to \infty} \frac{1-e^{-\alpha_{0,l}\pi_l}}{1-e^{-\alpha_{1,l}\pi_l}} = 1$ and $\lim_{\pi_l \to 0} \frac{1-e^{-\alpha_{0,l}\pi_l}}{1-e^{-\alpha_{1,l}\pi_l}} = \frac{\alpha_{0,l}}{\alpha_{1,l}}$. Hence, if

$$\frac{\alpha_{0,l}}{\alpha_{1,l}} > \frac{v(1)}{v(0)}$$

and π_l is small enough, we have

$$\frac{1 - e^{-\alpha_{0,l}\pi_l}}{1 - e^{-\alpha_{1,l}\pi_l}} > \frac{v(1)}{v(0)},$$

and so by Lemma 3 zero is in the support of both types of bidders. In this case, the payoff in the auction is $V_\theta^0 (\pi_h, \pi_l)$ and equilibrium must be socially optimal: $\pi_h = \pi_h^{opt}$, $\pi_l = \pi_l^{opt}$. This must be the case when c is sufficiently close to $\bar{c}$ (when $c = \bar{c}$, we have $\pi_l^{opt} = 0$). In the formal auction, by Proposition 2 the high type entry rate always satisfies $\pi_h^F > \pi_h^{opt}$, so that equilibrium is inefficient and generates a strictly lower total surplus than the informal auction. ∎

Proof of Proposition 4. As a first step, we show that with a given entry profile (π_h, π_l), a high type gets a strictly higher payoff in the formal auction than in the informal auction if π_l is small enough.

If zero is in the bidding support of both types in the informal auction, the result is immediate. Therefore, assume that there is no overlap in the bidding supports in the informal auction and denote by p' the common point in the two supports.

Let us contrast the informal auction to the formal auction, and consider the following bidding strategy. In the informal auction, let both types bid p'. Since this is in the support of both types, it generates the equilibrium payoff to both types. In the formal auction, let both types bid $\underline{p}(n) + \varepsilon$, where ε is small. Since we can take ε arbitrarily close to zero, we will for the rest of the proof calculate the winning probability and payoffs with this strategy in the limit $\varepsilon \to 0$. With this strategy, a bidder wins if and only if there are no (other) high type bidders, and the price in such a case is either zero or $\underline{p}(n)$. Clearly, this strategy gives the equilibrium payoff for both types (in the limit $\varepsilon \to 0$). Given these strategies, a player gets the same allocation in both auction formats (i.e. gets the object if and only if there

are no high types present), and therefore we may compare the players' payoffs across the auction formats by contrasting their expected payments conditional on winning.

Start with the low type. Since the low type gets expected payoff zero in both auction formats, she is indifferent between bidding p' in the informal auction and bidding $\underline{p}(n) + \varepsilon$ in the formal auction. Therefore, the expected payment conditional on winning must be the same in the two cases, leading to:

$$
\begin{aligned}
p' &= E\left(p \,|\, \theta = l, \text{ "win by bidding } \underline{p}(n) + \varepsilon\text{"}\right) \\
&= \Pr\left(N^l = 0 \,|\, \theta = l, N^h = 0\right) \cdot 0 \\
&\quad + \sum_{k=1}^{\infty} \Pr\left(N^l = k \,|\, \theta = l, N^h = 0\right) \underline{p}(k+1).
\end{aligned}
$$

Consider next the high type. Her expected payment when bidding $\underline{p}(n) + \varepsilon$ in formal auction is

$$
E\left(p \,|\, \theta = h, \text{ "win by bidding } \underline{p}(n) + \varepsilon\text{"}\right)
$$
$$
= \sum_{k=1}^{\infty} \Pr\left(N^l = k \,|\, \theta = h, N^h = 0\right) \underline{p}(k+1)
$$

and hence she prefers the formal auction if

$$
\sum_{k=1}^{\infty} \Pr\left(N^l = k \,|\, \theta = h, N^h = 0\right) \underline{p}(k+1)
$$
$$
< \quad p' = \sum_{k=1}^{\infty} \Pr\left(N^l = k \,|\, \theta = l, N^h = 0\right) \underline{p}(k+1), \tag{16}
$$

where

$$
\underline{p}(k+1) := E\left(v \,|\, \theta = l, N^h = 0, N^l = k\right)
$$

is a decreasing function of k (for $k \geq 1$). Note that the only difference in the two formulas in (16) is that the probability

$$
\Pr\left(N^l = k \,|\, \theta, N^h = 0\right)
$$

is conditioned on $\theta = h$ in the first line and $\theta = l$ in the second line. Since a high signal makes state $\omega = 1$ more likely, a simple sufficient condition for (16) to hold is that

$$
\Pr\left(N^l = k \,|\, \omega = 1, N^h = 0\right) < \Pr\left(N^l = k \,|\, \omega = 0, N^h = 0\right)
$$

for all $k \geq 1$. Since

$$
N^l \,|\, \left(\omega = 1, N^h = 0\right) \sim Poisson\left(\lambda_{\omega,l}\right),
$$

we know that (16) holds if

$$\lambda_{0,l} e^{-\lambda_{0,l}} > \lambda_{1,l} e^{-\lambda_{1,l}},$$

that is

$$\alpha_{0,l} \pi_l \, e^{-\alpha_{0,l} \pi_l} > \alpha_{1,l} \pi_l \, e^{-\alpha_{1,l} \pi_l}$$

or

$$\pi_l < \frac{\log \alpha_{0,l} - \log \alpha_{1,l}}{\alpha_{0,l} - a_{1,l}}.$$

Therefore, a high-type bidder has a lower expected payment in the formal auction for π_l small enough, which means that her expected payoff is higher in that auction formal.

The second step is just to conclude that changing the auction format from the informal one to the formal one will increase distortion. Fixing the equilibrium entry profile $\left(\pi_h^{IF}, \pi_l^{IF}\right)$ of the informal auction, but changing the auction format to the formal auction, a high type bidder gets in the bidding equilibrium strictly more than c. To compensate this, the high-type entry rate in an equilibrium to the formal auction must satisfy $\pi_h^F > \pi_h^{IF}$. Since in both auction formats $\pi_l = \pi_l^* (\pi_h)$, the formal auction hence moves the equilibrium entry profile along $\pi_l^* (\pi_h)$ further away from the social optimum than the informal auction. By Lemma 4, the total surplus is then higher in the informal auction than in the formal auction. ∎

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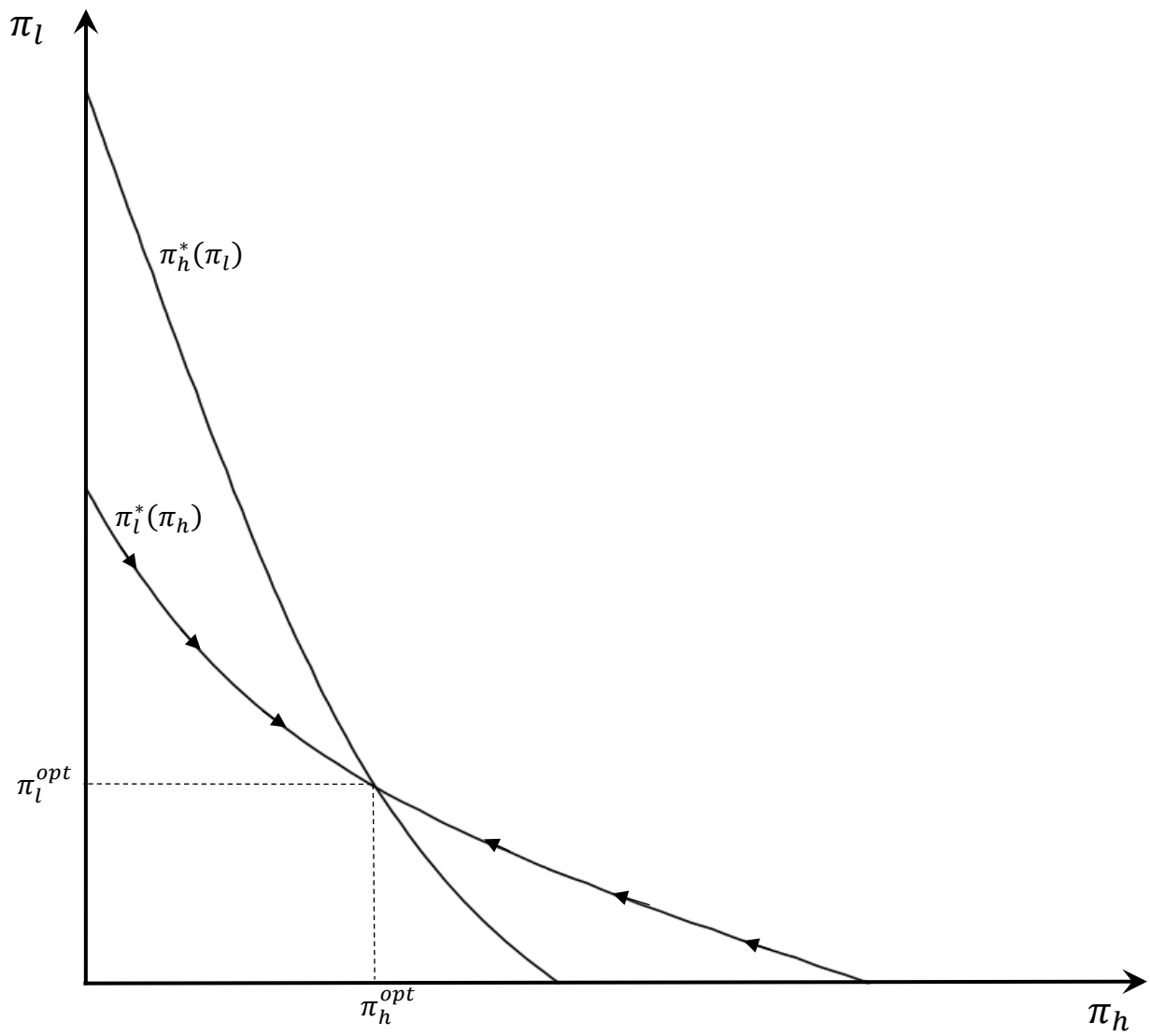SWEETING, A., AND V. BHATTACHARYA (2015): "Selective entry and auction design," *International Journal of Industrial Organization*, 43, 189–207.

Figure 1: Social planner's reaction curves.