E   Supplementary Online Appendix for Gradual Learning from Incremental Actions - not for publication

E.1   Implementation with posted prices

Proposition 5 in the main text defines a posted price $P(x, q)$ that implements an arbitrary boundary policy Q. At the boundary $\widetilde{x}$, this function is given by equation (11) of the main paper, which we reproduce for convenience:

$$P\left(\widetilde{x}\left(q\right), q\right) = \widetilde{x}(q)(v_H(\theta(q)) + (1 - \widetilde{x}(q))v_L(\theta(q)))$$

$$- \mathbb{E}\left[\int_{\underline{\theta}}^{\theta(q)} e^{-r(\tau(s) - \tau(\theta(q)))} \left(\widetilde{x}_{\tau(s)} v'_H(s) + (1 - \widetilde{x}_{\tau(s)})v'_L(s)\right) ds \,\middle|\, x(q), q\right]. \tag{33}$$

Here we ask if we can implement the same boundary policy Q using posted price instruments that only depend on one state variable, either stock q or belief x. In order for the agents to have the right stopping incentive along the intended stopping boundary, such a transfer must coincide with the posted price in Proposition 5 at the boundary, but will differ below it. The key question is whether deviations to stop below the boundary can be prevented.

By a *dynamic posted price*, we refer to a transfer that is a function of the current belief and hence continuously responds to news about the state. The dynamic posted price, $P^D(x)$, is pinned down by (33) at the boundary points:

$$P^D(x) := \begin{cases} P(x, \widetilde{q}(x)) & \text{for } x \geq \widetilde{x}(0) \\ P(\widetilde{x}(0), 0) & \text{for } x < \widetilde{x}(0) \end{cases} \tag{34}$$

where $\widetilde{q}(x) : [\widetilde{x}(0), 1] \longrightarrow [0, q_1]$ is the inverse of $\widetilde{x}(\cdot)$. If agent θ stops at state (x, q), his stopping payoff is

$$u_\theta^D(x) = x v_H(\theta) + (1 - x) v_L(\theta) - P^D(x).$$

The term $P^D(x)$ makes the optimal stopping problem more complicated than the corresponding problem without a transfer. Yet, without even explicitly solving the individual agents' stopping problems we will prove that this pricing rule is immune to deviations to stop below the intended boundary.

When using the dynamic posted price rule (34), the designer needs to continuously observe the news process and carry out detailed Bayesian calculation to adjust the transfer. To make the job of the designer easier and the commitment assumption more palatable we consider as an alternative *simple posted prices* that depend only on the stock q. As with the dynamic posted price, we set the simple posted price $P^S(q)$ to coincide with (33) at boundary points:

$$P^S(q) := P(\tilde{x}(q), q) \text{ for } q \in [0, q_1]. \tag{35}$$

Now agents face an optimal stopping problem where the stopping payoff depends on q as well as x:

$$u_\theta^S(x, q) = x v_H(\theta) + (1-x) v_L(\theta) - P^S(q).$$

An important property of this scheme is that the stopping payoff changes abruptly at the boundary. Whether the transfer is increasing or decreasing turns out to be critical for providing sufficient stopping incentive at the boundary without inviting deviations to stop below it. If the transfer is increasing, even a slight postponement would make stopping more expensive for the agent at the boundary, providing an additional incentive to stop at the boundary relative to the states below it. However, if the transfer is decreasing, stopping at the boundary is less attractive as a delay would be rewarded with a reduction in transfer payment. As a result, if the decreasing simple posted price scheme is such that an agent is willing to stop at the boundary, he wants to stop even before reaching it.

To show that the monotonicity condition for the simple transfer is substantial, we will give an example in the next subsection where $P^S(q)$ is not monotonic.

We prove below:

Proposition 6. *Let Q denote a boundary policy with a strictly increasing policy function $\tilde{x}$ and $\tilde{x}(q_1) = 1$. Then:*

- *Q can be implemented by a dynamic posted price $P^D(x)$ in (34).*

- *Q can be implemented by a simple posted price $P^S(q)$ in (35) if and only if $\left(P^S\right)'(q) \geq 0$ for all $q \in [0, q_1]$.*

Proof. Fix a boundary policy Q with a strictly increasing boundary $\tilde{x}(q)$ and $\tilde{x}(q_1) = 1$. We analyze the optimal stopping problem of arbitrary type θ. We will show that under $P^D(x)$ it is optimal to stop at the first hitting time of the point $(\tilde{x}(q(\theta)), q(\theta))$ and the same conclusion holds under $P^S(q)$ if $\left(P^S\right)'(q) \geq 0$ for all $0 \leq q \leq q_1$. Finally, we show that if $\left(P^S\right)'(q) < 0$ for some q, then type $\theta(q)$ has a profitable deviation to stopping at some (x', q') with $x' < \tilde{x}(q')$, $q' < q$.

Preliminary step: Optimal stopping when stopping below boundary prohibited

As a preliminary step we just note that under both $P^D(x)$ and $P^S(q)$ it is optimal for θ to stop at the first hitting time of $(\tilde{x}(q(\theta)), q(\theta))$ if stopping below the boundary is prohibited. The restricted stopping problem where stopping is only allowed on the boundary is still a Markovian stopping problem where the optimal solution is a first-hitting time of some of the boundary points. Under both $P^D(x)$ and $P^S(x)$, stopping at boundary point $(\tilde{x}(q), q)$ entails transfer payment $P(\tilde{x}(q), q)$ given in (33), which is designed in such a way that θ maximizes her ex-ante expected payoff by stopping as soon as $(\tilde{x}(q(\theta)), q(\theta))$ is hit (see proposition 5 and its proof in the main paper).

Part 1: Optimal stopping under dynamic posted price $P^D(x)$

We will utilize the property that the stopping value $u_\theta^D(x)$ is indepenent of q to show that even if stopping below the boundary is allowed, an agent will optimally stop at a first-hitting time of *some* boundary point. For contradiction, assume that type θ stops with a strictly positive probability at the first-hitting time of some point below the boundary and let (x', q') be the "left-most" such point. Formally, denoting by $F_\theta(x, q)$ the optimal value function of type θ, let (x', q'), $x' < \tilde{x}(q')$, be a state point such that $F_\theta(x', q') = u_\theta^D(x')$ and $F_\theta(x, q) > u_\theta^D(x)$ for all points with $q < q'$.[14] We can write $F_\theta(x, q) = B_\theta(q) \Phi(x, q)$ for some function $B_\theta(q)$ so

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[14] Note that we must have $F_\theta(\tilde{x}(q), q) > u_\theta^D(\tilde{x}(q))$ also at all boundary points with $q < q'$ because all those boundary points are visited before point (x', q') and hence otherwise stopping could not take place with positive probability at (x', q') .

that

$$\frac{\partial}{\partial q} F_\theta(x, q) = B'_\theta(q) \Phi(x, q) + B_\theta(q) \Phi_q(x, q)$$

$$= \Phi(x, q) \left[B'_\theta(q) + B_\theta(q) \beta'(q) \ln\left(\frac{x}{1-x}\right) \right],$$

where we have used that $\Phi_q(x, q) = \beta'(q) \ln\left(\frac{x}{1-x}\right) \Phi(x, q)$.

Similar to the proof of Proposition 1, we note that the partial of a value function w.r.t. q must be zero at the boundary $x = \widetilde{x}(q)$, i.e. for $q < q'$ we have $\frac{\partial}{\partial q} F_\theta(x, q)_{|x=\widetilde{x}(q)} = 0$ and so

$$B'_\theta(q) + B_\theta(q) \beta'(q) \ln\left(\frac{\widetilde{x}(q)}{1 - \widetilde{x}(q)}\right) = 0.$$

But since $\beta'(q) < 0$ and $\widetilde{x}(q) > x$, this implies that

$$B'_\theta(q) + B_\theta(q) \beta'(q) \ln\left(\frac{x}{1-x}\right) > 0,$$

and so we have $\frac{\partial}{\partial q} F_\theta(x, q) > 0$. This is a contradiction with our assumption that $F_\theta(x', q') = u^D_\theta(x')$ and $F_\theta(x, q) > u^D_\theta(x)$ for all points with $q < q'$.

We can conclude that the optimal stopping time must be the first hitting time of some boundary point. This means that the solution must be the same as if stopping below the bounday is prohibited. By step 1 above, it is then optimal for θ to stop at the first-hitting time of point $(\widetilde{x}(q(\theta)), q(\theta))$.

Part 2: Optimal stopping under simple posted price $P^S(q)$

Since the stopping value $u^S_\theta(x, q)$ is now a function of q as well as x, the argument in the second step above does not hold and we will use a more direct approach. We will first directly show that if $P^S(q)$ is increasing everywhere, it is optimal for type θ to stop at the first hitting time of point $(\widetilde{x}(q(\theta)), q(\theta))$ that we denote by

$$\tau^*_\theta := \inf\left\{ t : (x_t, q_t) = (\widetilde{x}(q(\theta)), q(\theta)) \right\}.$$

After that we will show that if $\left(P^S\right)'(q) < 0$ for some q, then type $\theta(q)$ has a profitable deviation to stop at some (x', q') with $q' < q(\theta)$, $x' < \widetilde{x}(q')$.

Let us first investigate the implication of the condition $\left(P^S\right)'(q) \geq 0$. We can write

$$P^S(q) = x(q) v_H(\theta(q)) + (1 - x(q)) v_L(\theta(q)) - S(q),$$

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where $S(q)$ is the information rent obtained by type $\theta(q)$ evaluated at the moment of stopping:

$$S(q) := \mathbb{E}\left[\int_{\underline{\theta}}^{\theta(q)} e^{-r\tau(s)} \left(x_{\tau(s)} v'_H(s) + \left(1 - x_{\tau(s)}\right) v'_L(s)\right) ds \,|\, \widetilde{x}(q), q\right].$$

We next differentiate $P^S(q)$ with respect to q. Note first that for any $s < \theta(q)$ we can write

$$\mathbb{E}\left[e^{-r\tau(s)} \,|\, \widetilde{x}(q), q\right] = A_s(q) \Phi(\widetilde{x}(q), q) \tag{36}$$

for some function $A_s(q)$. Since q_t increases at the boundary, the partial derivative of (36) w.r.t. q must be zero there, i.e.: $\frac{\partial}{\partial q}\mathbb{E}\left[e^{-r\tau(s)} \,|\, \widetilde{x}(q), q\right] = 0$. Therefore, the change of (36) when moving the initial point along the boundary is

$$
\begin{aligned}
\frac{d}{dq}\mathbb{E}\left[e^{-r\tau(s)} \,|\, \widetilde{x}(q), q\right] &= \frac{\partial}{\partial x}\mathbb{E}\left[e^{-r\tau(s)} \,|\, x, q\right]_{x=\widetilde{x}(q)} \widetilde{x}'(q) \\
&= A_s(q) \Phi_x(\widetilde{x}(q), q) \widetilde{x}'(q) \\
&= \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)} \widetilde{x}'(q) \mathbb{E}\left[e^{-r\tau(s)} \,|\, \widetilde{x}(q), q\right]. \tag{37}
\end{aligned}
$$

Using this, we get:

$$
\begin{aligned}
S'(q) = &\left(\widetilde{x}(q) v'_H(\theta(q)) + (1 - \widetilde{x}(q)) v'_L(\theta(q))\right)\theta'(q) \\
&+ \int_{\underline{\theta}}^{\theta(q)} \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)} \widetilde{x}'(q) \mathbb{E}\left[e^{-r\tau(s)} \left(\widetilde{x}_{\tau(s)} v'_H(s) + \left(1 - \widetilde{x}_{\tau(s)}\right) v'_L(s)\right) |\, \widetilde{x}(q), q\right] ds \\
= &\left(\widetilde{x}(q) v'_H(\theta(q)) + (1 - \widetilde{x}(q)) v'_L(\theta(q))\right)\theta'(q) + \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)} S(q) \widetilde{x}'(q), \tag{38}
\end{aligned}
$$

where the first term is from differentiation with respect to the integral bound and the second term is from differentiation with respect to the initial point $\widetilde{x}(q)$ using (37). The derivative of $P^S(q)$ with respect to q is then

$$
\begin{aligned}
\left(P^S\right)'(q) = &\widetilde{x}'(q)\left(v_H(\theta(q)) - v_L(\theta(q))\right) \\
&+ \left[\widetilde{x}(q) v'_H(\theta(q)) + (1 - \widetilde{x}(q)) v'_L(\theta(q))\right]\theta'(q) - S'(q) \\
= &\left[v_H(\theta(q)) - v_L(\theta(q)) - \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)} S(q)\right] \widetilde{x}'(q).
\end{aligned}
$$

Hence, $\left(P^S\right)'(q) \geq 0$ is equivalent to

$$v_H(\theta(q)) - v_L(\theta(q)) \geq \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)} S(q), \tag{39}$$

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a condition that we will utilize below.

We now show that it cannot be optimal to stop below the boundary $\widetilde{x}(q)$ for $q < q(\theta)$. Fix (x, q) with $q < q(\theta)$ and $x < \widetilde{x}(q)$. We show that stopping at (x, q) is dominated by waiting until x hits $\widetilde{x}(q)$. Denote the value under that stopping rule by

$$\underline{F}_\theta(x, q) := \mathbb{E}\left[e^{-r\tau(\widetilde{x}(q))}u_\theta^S(x, q)\,|x, q\right],$$

where $\tau(\widetilde{x}(q)) = \inf\{t : x_t = \widetilde{x}(q)\}$. Writing $\underline{F}_\theta(x, q) := \underline{A}_\theta(q)\Phi(x, q)$ and solving $\underline{A}_\theta(q)$ from the boundary condition $\underline{F}_\theta(\widetilde{x}(q), q) = u_\theta^S(\widetilde{x}(q), q)$ gives us

$$\underline{F}_\theta(x, q) = \frac{\Phi(x, q)}{\Phi(\widetilde{x}(q), q)}u_\theta^S(\widetilde{x}(q), q).$$

We aim to show that below the boundary, i.e. for $x < \widetilde{x}(q)$, we have $\underline{F}_\theta(x, q) > u_\theta^S(x, q)$. Let us differentiate these functions with respect to x, and evaluate the derivative at the boundary:

$$\frac{\partial}{\partial x}[u_\theta(x, q)]_{x=\widetilde{x}(q)} = v_H(\theta) - v_L(\theta)$$

and

$$\frac{\partial}{\partial x}[\underline{F}_\theta(x, q)]_{x=\widetilde{x}(q)} = \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)}u_\theta(\widetilde{x}(q), q)$$

$$= \frac{\Phi_x(\widetilde{x}(q), q)}{\Phi(\widetilde{x}(q), q)}[\widetilde{x}(q)(v_H(\theta) - v_H(\theta(q))) + (1 - \widetilde{x}(q))(v_L(\theta) - v_L(\theta(q))) + S(q)]$$

$$\leq v_H(\theta)\frac{\beta(q) - \widetilde{x}(q)}{1 - \widetilde{x}(q)} + v_L(\theta)\frac{\beta(q) - \widetilde{x}(q)}{\widetilde{x}(q)} - v_H(\theta(q))\frac{\beta(q) - 1}{1 - \widetilde{x}(q)} - v_L(\theta(q))\frac{\beta(q)}{\widetilde{x}(q)}$$

$$= v_H(\theta) - v_L(\theta) + \left(\frac{\beta(q) - 1}{1 - \widetilde{x}(q)}\right)(v_H(\theta) - v_H(\theta(q))) + \frac{\beta(q)}{\widetilde{x}(q)}(v_L(\theta) - v_L(\theta(q)))$$

$$< v_H(\theta) - v_L(\theta),$$

where the first inequality utilizes the fact that $(P^s)'(q) \geq 0$ is equivalent to (39) and the second inequality utilizes the fact that $q < q(\theta)$ implies that $v_H(\theta) - v_H(\theta(q)) \leq 0$ and $v_L(\theta) - v_L(\theta(q)) \leq 0$ with at least one of the inequalities strict It now follows that

$$\frac{\partial}{\partial x}[\underline{F}_\theta(x, q)]_{x=\widetilde{x}(q)} < \frac{\partial}{\partial x}\left[u_\theta^S(x, q)\right]_{x=\widetilde{x}(q)},$$

and since $\underline{F}_\theta(x, q)$ is strictly convex in x and positive for $x < \widetilde{x}(q)$ while $u_\theta^S(x, q)$ is linear in x, we have

$$\underline{F}_\theta(x, q) > u_\theta^S(x, q) \text{ for } x < \widetilde{x}(q),$$

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and so stopping at (x, q) is strictly dominated by waiting until x hits $\tilde{x}(q)$. Since (x, q) was arbitrarily chosen, it can never be optimal to stop strictly below the boundary for $q < q(\theta)$. But we know from Step 1 of the proof that stopping at point $(\tilde{x}(q(\theta)), q(\theta))$ dominates stopping at other boundary points. This implies that it can never be optimal for θ to stop ealier than at τ_θ^*.

The above argument ruled out stopping below the boundary only for $q < q(\theta)$. It remains to show that θ cannot benefit from delaying stopping beyond time τ_θ^* to the hitting time of some (x, q) with $q > q(\theta)$, $x < \tilde{x}(q)$. For that it suffices to show that it is optimal to stop at all boundary points for $q \geq q(\theta)$, i.e. at all $(\tilde{x}(q), q)$ for $q \geq q(\theta)$. The proof follows similar reasoning as the proof of Proposition 1.

Suppose, to the contrary, that there is some $(\tilde{x}(q), q)$ such that $q \geq q(\theta)$ where it is not optimal to stop. At that point we therefore have $\tilde{F}_\theta(\tilde{x}(q), q) > u_\theta^S(\tilde{x}(q), q)$, where $\tilde{F}_\theta(x, q)$ is the value function under the optimal stopping rule (whatever that may be). We will show next that this implies that along the boundary, the rate of change in $\tilde{F}_\theta(\tilde{x}(q), q)$ is higher than in $u_\theta^S(\tilde{x}(q), q)$:

$$\frac{d}{dq}\tilde{F}_\theta(\tilde{x}(q), q) > \frac{d}{dq}u_\theta^S(\tilde{x}(q), q). \tag{40}$$

We prove this separately in two possible cases. First, suppose that it is optimal to stop for some (x', q), where $x' < \tilde{x}(q)$. In that case $\tilde{F}_\theta(x', q) = u_\theta^S(x', q)$. Since $\tilde{F}_\theta(x, q)$ must be strictly convex in x whenever it is optimal to wait (i.e. when $\tilde{F}_\theta(x, q) > u_\theta^S(x, q)$), whereas $u_\theta^S(x, q)$ is linear in x with slope $v_H(\theta) - v_L(\theta)$, we must have

$$\frac{\partial}{\partial x}\left[\tilde{F}_\theta(x, q)\right]_{x=\tilde{x}(q)} > v_H(\theta) - v_L(\theta). \tag{41}$$

Let us now compare the rates of change in $\tilde{F}_\theta(\tilde{x}(q), q)$ and $u_\theta^S(\tilde{x}(q), q)$ along the boundary. We have

$$\begin{aligned}
\frac{d}{dq}\tilde{F}_\theta(\tilde{x}(q), q) &= \frac{\partial}{\partial x}\left[\tilde{F}_\theta(x, q)\right]_{x=\tilde{x}(q)}\tilde{x}'(q) + \frac{\partial}{\partial q}\tilde{F}_\theta(\tilde{x}(q), q) \\
&= \frac{\partial}{\partial x}\left[\tilde{F}_\theta(\tilde{x}, q)\right]_{x=\tilde{x}(q)}\tilde{x}'(q),
\end{aligned}$$

where we have again utilized the fact that the partial of a waiting value w.r.t q must vanish at the boundary where q is increased, i.e. $\frac{\partial}{\partial q}\tilde{F}_\theta(\tilde{x}(q), q) = 0$.

The stopping value at (x, q) is

$$
\begin{aligned}
u_\theta^S (x, q) &= x v_H (\theta) + (1 - x) v_L (\theta) - P^S (q) \\
&= x v_H (\theta) + (1 - x) v_L (\theta) \\
&\quad - x (q) v_H (\theta (q)) - (1 - x (q)) v_L (\theta (q)) + S (q)
\end{aligned}
$$

and so at the boundary $x = \tilde{x} (q)$ the stopping value is

$$
\begin{aligned}
u_\theta^S (\tilde{x} (q), q) &= \tilde{x} (q) (v_H (\theta) - v_H (\theta (q))) \\
&\quad + (1 - \tilde{x} (q)) (v_L (\theta) - v_L (\theta (q))) + S (q)
\end{aligned}
$$

and we can compute its rate of change along the boundary as:

$$
\frac{d}{dq} u_\theta^S (\tilde{x} (q), q) = \tilde{x}' (q) (v_H (\theta) - v_H (\theta (q)) - v_L (\theta) + v_L (\theta (q)))
$$

$$
- [\tilde{x} (q) v_H' (\theta (q)) + (1 - \tilde{x} (q)) v_L' (\theta (q))] \theta' (q) + S' (q)
$$

$$
= \tilde{x}' (q) \left[(v_H (\theta) - v_L (\theta)) - (v_H (\theta (q)) - v_L (\theta (q))) + \frac{\Phi_x (\tilde{x} (q), q)}{\Phi (\tilde{x} (q), q)} S (q) \right], \quad (42)
$$

where the latter equality uses (38). Using (41) and (39), we then see that (40) holds.

We move to the second case. Suppose that it is optimal to wait for all (x, q), where $x < \tilde{x} (q)$, in which case $\widetilde{F}_\theta (x, q) > u_\theta^S (x, q)$ for all $x < \tilde{x} (q)$ and $\widetilde{F}_\theta (0, q) = 0$. In that case, function $\widetilde{F}_\theta (x, q)$ must take the form

$$
\widetilde{F}_\theta (x, q) = \widehat{A}_\theta (q) \Phi (x, q)
$$

for some function $\widehat{A}_\theta (q)$. Our assumption $\widetilde{F}_\theta (\tilde{x} (q), q) > u_\theta^S (\tilde{x} (q), q)$ is equivalent to

$$
\widehat{A}_\theta (q) \Phi (\tilde{x} (q), q) > \tilde{x} (q) (v_H (\theta) - v_H (\theta (q))) + (1 - \tilde{x} (q)) (v_L (\theta) - v_L (\theta (q))) + S (q)
$$

or

$$
\widehat{A}_\theta (q) > \frac{\tilde{x} (q) (v_H (\theta) - v_H (\theta (q))) + (1 - \tilde{x} (q)) (v_L (\theta) - v_L (\theta (q))) + S (q)}{\Phi (\tilde{x} (q), q)}.
$$

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This implies

$$\frac{d}{dq}\widetilde{F}_\theta\left(\widetilde{x}\left(q\right),q\right) > \widetilde{x}'\left(q\right)\frac{\Phi_x\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}$$
$$\times\left[\widetilde{x}\left(q\right)\left(v_H\left(\theta\right) - v_H\left(\theta\left(q\right)\right)\right) + \left(1 - \widetilde{x}\left(q\right)\right)\left(v_L\left(\theta\right) - v_L\left(\theta\left(q\right)\right)\right) + S\left(q\right)\right]$$
$$= \widetilde{x}'\left(q\right)\left[\frac{\beta\left(q\right) - \widetilde{x}\left(q\right)}{\left(1 - \widetilde{x}\left(q\right)\right)}\left(v_H\left(\theta\right) - v_H\left(\theta\left(q\right)\right)\right)\right.$$
$$\left. + \frac{\beta\left(q\right) - \widetilde{x}\left(q\right)}{\widetilde{x}\left(q\right)}\left(v_L\left(\theta\right) - v_L\left(\theta\left(q\right)\right)\right) + \frac{\Phi_x\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}S\left(q\right)\right].$$

Noting that $q > q\left(\theta\right)$ means that $v_H\left(\theta\right) - v_H\left(\theta\left(q\right)\right) \geq 0$ and $v_L\left(\theta\right) - v_L\left(\theta\left(q\right)\right) \geq 0$, and comparing the expression above to (42), we note again that (40) holds.

Having proved that $\widetilde{F}_\theta\left(\widetilde{x}\left(q\right),q\right) > u_\theta\left(\widetilde{x}\left(q\right),q\right)$ implies (40), we note that this would imply that $\widetilde{F}_\theta\left(\widetilde{x}\left(q'\right),q'\right) > u_\theta\left(\widetilde{x}\left(q'\right),q'\right)$ along the boundary $\left(\widetilde{x}\left(q'\right),q'\right)$ for all $q \leq q' \leq q_1$, and so $\widetilde{F}_\theta\left(\widetilde{x}\left(q_1\right),q_1\right) > u_\theta\left(\widetilde{x}\left(q_1\right),q_1\right)$. This is a contradiction since we know that it must be optimal to stop at state point $\left(\widetilde{x}\left(q_1\right),q_1\right)$. We conclude that it is optimal to stop at all $\left(\widetilde{x}\left(q\right),q\right)$, $q\left(\theta\right) \leq q \leq q_1$. It now follows that the optimal stopping time for θ is τ_θ^*, i.e. the first hitting time of $\left(\widetilde{x}\left(q\left(\theta\right)\right),q\left(\theta\right)\right)$.

As a final point we note that our conclusion hinges critically on the assumption that $\left(P^S\right)'\left(q\right) \geq 0$ for all q. If, in contrast, $\left(P^S\right)'\left(q\right) < 0$ for some $q \in \left(0,q_1\right)$, then there is a profitable deviation for type $\theta = \theta\left(q\right)$ to stop ealier than at time τ_θ^*. To see this, note that $\left(P^S\right)'\left(q\right) < 0$ implies that for $\theta = \theta\left(q\right)$, we have

$$v_H\left(\theta\left(q\right)\right) - v_L\left(\theta\left(q\right)\right) < \frac{\Phi_x\left(\widetilde{x}\left(q\right),q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}S\left(q\right). \tag{43}$$

Consider then the value of type $\theta\left(q\right)$ who plans to stop at $\left(\widetilde{x}\left(q\right),q\right)$:

$$F_{\theta\left(q\right)}\left(x,q\right) = \widetilde{A}_{\theta\left(q\right)}\left(q\right)\Phi\left(x;q\right).$$

Since

$$F_{\theta\left(q\right)}\left(\widetilde{x}\left(q\right),q\right) = u_{\theta\left(q\right)}^S\left(\widetilde{x}\left(q\right),q\right)$$
$$= v_H\left(\theta\left(q\right)\right) - v_L\left(\theta\left(q\right)\right) - \left(v_H\left(\theta\left(q\right)\right) - v_L\left(\theta\left(q\right)\right) - S\left(q\right)\right) = S\left(q\right),$$

we have

$$\widetilde{A}_{\theta\left(q\right)}\left(q\right) = \frac{S\left(q\right)}{\Phi\left(\widetilde{x}\left(q\right),q\right)}$$

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and so

$$\frac{\partial}{\partial x}\left[F_{\theta(q)}(x,q)\right]_{x=\widetilde{x}(q)} = \frac{\Phi_x\left(\widetilde{x}(q),q\right)}{\Phi\left(\widetilde{x}(q),q\right)}S(q).$$

Noting that

$$\frac{\partial}{\partial x}\left[u_{\theta(q)}^S(x,q)\right]_{x=\widetilde{x}(q)} = v_H\left(\theta(q)\right) - v_L\left(\theta(q)\right),$$

equation (43) implies

$$\frac{\partial}{\partial x}\left[u_{\theta(q)}^S(x,q)\right]_{x=\widetilde{x}(q)} < \frac{\partial}{\partial x}\left[F_{\theta(q)}(x;q)\right]_{x=\widetilde{x}(q)}$$

and since $u_{\theta(q)}^S\left(\widetilde{x}(q),q\right) = F_{\theta(q)}\left(\widetilde{x}(q),q\right)$, it follows that for some $x < \widetilde{x}(q)$ we have

$$u_{\theta(q)}^S(x,q) > F_{\theta(q)}(x,q).$$

By continuity of F and u, this implies that there is some $q' < q$, $x' < \widetilde{x}(q')$, such that $u_{\theta(q)}^S(x',q') > F_{\theta(q)}(x',q')$, and hence type $\theta(q)$ has a strictly beneficial deviation to stopping at that state, and that state is reached before τ_θ^* with a strictly positive probability.

We have now shown that $\widetilde{x}(q)$ can be implemented by the simple posted price $P^s(q)$ if an only if $(P^s)'(q) \geq 0$ for all $0 \leq q \leq q_1$ and the proof is complete.

□

E.2 Socially optimal transfers

We demonstrate here the properties of socially optimal transfers and note that implementation is not always possible using *simple posted prices*. The left panel of Figure 7 depicts the socially optimal boundary policy $\widetilde{x}$ for stopping payoffs $v_H(q) = 1 - q$, $v_L(q) = -\eta - q$, where the three cases correspond to different values for parameter η which increases the cost of stopping in the low state. The right panel shows the corresponding socially optimal transfer function $P(\widetilde{x}(q),q)$ as a function of q. We see that $P(\widetilde{x}(q),q)$ is always negative: the designer pays each agent so that they internalize the information generation effect. The optimal transfer goes smoothly to zero when q approaches 1 because the information generation effect disappears and the decentralized and the optimal policies coincide

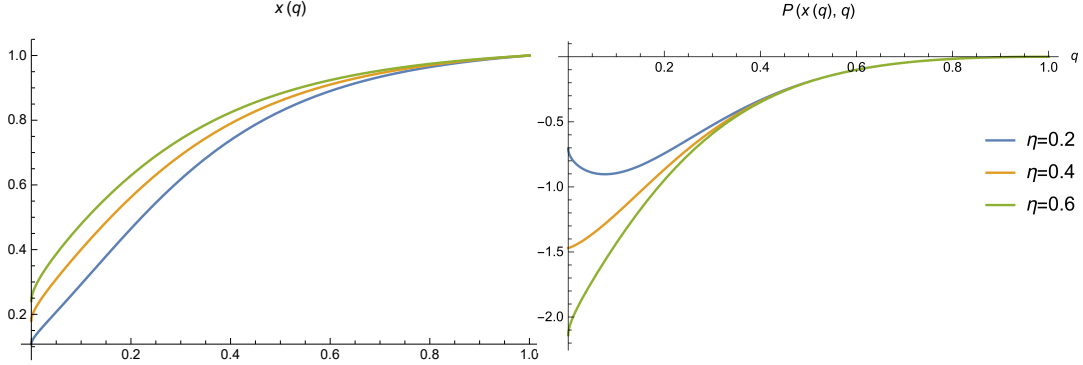even without transfers. These properties hold generally for the socially optimal transfer.

Figure 7: The left panel shows the socially optimal policy for different values of η. The right panel shows the corresponding transfers. $v_H(q) = 1 - q, v_L(q) = -\eta - q, r = 0.1, \sigma = 1$.

We know from Proposition 6 that we can always use a dynamic posted price to implement the policy, but a simple posted price works only if $P^S(q) := P(\tilde{x}(q), q)$ is monotone. In this example, the optimal policy can only be implemented with simple posted prices when the risk parameter is large enough. That is, transfers are non-monotone when $\eta = 0.2$, but motone when $\eta = 0.4$ or $\eta = 0.6$. The transfer rule becomes non-monotone When η is small because the highest types are almost willing to stop even without transfers whereas lower types need larger transfers as they get a large negative payoff if $\omega = L$ and need to be compensated for the option value of waiting.

E.3 Durable goods: additional result

We show here that the properties for the competitive and monopoly solutions described in section 4.3 of the main paper hold generally.

Proposition 7. *There exist cutoffs $x_a \in (0,1)$ and $x_b \in (0,1)$ such that the monopoly quantity is larger if the initial belief is below x_a and the competitive market quantity is larger if the initial belief is above x_b.*

Proof. The existence of $x_b < 1$ follows from that the continuity of the policy

functions and form that the complete information quantity is larger in the competitive market: $q^C(1) > q^M(1)$, where $q^C(1)$ solves $\theta(q^C(1)) = c$ and $q^M(1)$ solves $\theta(q^M(1)) - (1 - F(\theta(q^M(1))))/f(\theta(q^M(1))) = c$.

The existence of $x_a > 0$ follows from the same argument as that socially optimal policy is below the decentralized policy (details omitted): 1) To see that $x^M(0) \neq x^C(0)$, observe that the smooth pasting and value matching conditions for both the decentralized and the planner's policies cannot hold simultaneously when we approach $q \to 0$ along x^M. The reason why the same proof works for the monopolist's policy as for the planner's policy is that the monopolist's flow payoff is the same as the social planner's when $q = 0$. 2) To rule out $x^M(0) > x^C(0)$, notice that the monopolist gets strictly positive profits by selling to some small q whenever the initial belief is above $x^C(0) = x^{stat}(0)$. 3) Now, it is enough to use the continuity and monotonicity of the policy functions the same way as in the proof of Proposition 3 in the main text to conclude that there exists $x_a > 0$ such that the monopolists sells more for all beliefs below x_a.

$\square$

E.4 Type-dependent informativeness: experts versus fanatics

Our baseline model assumes that all agents are equally informative: one unit of the stock q produces the same (marginal) amount of information. However, this might not necessarily be true in many applications of our model. For example, first buyers might be fans of the product whose experience matters less for the general population. Or conversely, the first units might be acquired by experts who are able to deduce the true value of the product much more quickly than average users. In this subsection, we show that as long as we can ensure that agents' stopping decision are monotone so that higher types stop first, the analysis in the baseline model is still valid.

Let the marginal informativeness of agent θ be $i(\theta) > 0$ so that the total "information stock" at time t is $z_t := \int_{\theta \in S_t} i(\theta) dF(\theta)$ where S_t is the set of agents

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who have stopped by time t. The evolution of the news process Y_t is then given by $dy_t = z_t \mu_\omega dt + \sigma \sqrt{z_t} dw_t$. The two especially interesting cases are when the high types are more informative, $i'(\theta) > 0$ ("experts"), and when the high types are less informative, $i'(\theta) < 0$ ("fanatics"). Throughout we assume that function i is continuously differentiable.

Notice first that the stopping profile in the decentralized equilibrium must be monotone in type because individual agents ignore the effect their stopping has on information, and thus Lemma 1 holds as in the baseline model. Because of this, we have a one-to-one relationship between the stock and the information stock: $z_t = h(q_t)$ where h is an "informativeness" function that satisfies $h'(q) = i(\theta(q))$. The analysis of the decentralized equilibrium then stays essentially the same as before.[15]

The socially optimal stopping profile is monotone in the experts environment as both the marginal informativeness and the stopping payoff are increasing in the type. It is also monotone in the fanatics environments when marginal informativeness is not changing too extremely. In these cases, we can solve the socially optimal solution with a change of variables. Consider a problem that is otherwise identical to the problem solved in Section 3.5 but where the stock process Q is replaced with the information stock process Z and where the stopping payoffs are scaled so that they take into account how many agents need to stop to increase the information stock by one unit: $\hat{v}_\omega(z) := v_\omega(h^{-1}(z))h^{-1'}(z)$. We show here that solving this problem solves the planner's original problem when informativeness of stopping is type-dependent:

Proposition 8. *Suppose either condition (i) or condition (ii) holds:*

(i) *Marginal informativeness is increasing in type (experts).*

(ii) *Marginal informativeness is decreasing in type (fanatics) but informativeness does not change too extremely: $\frac{-i'(\theta)}{i(\theta)} \leq \frac{-v_L'(\theta)}{v_L(\theta)}$ holds for all θ.*

Then the socially optimal policy is characterized by $x^*(z(q))$ where x^* is as defined

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[15]We only need to adjust β function (plug in $h(q)$ instead of q). With this change Proposition 1 still characterizes decentralized behavior.

in Proposition 2.

Proof. **Part (ii): fanatics.** Suppose $-i'(\theta)/i(\theta) \in [0, -v'_L(\theta)/v_L(\theta)]$.

First, we argue that the socially optimal stopping profile is monotone. We can use the same argument as in Lemma 2 but we need to normalize the informativeness of different agents. We show that for all agents $\theta, \theta' \in [\underline{\theta}, \overline{\theta}]$ such that $\theta > \theta'$ and for all realized stopping times $t, t' \in \mathbb{R}_+$ such that $t \leq t'$,

$$e^{-rt}\frac{v_\omega(\theta)}{i(\theta)} + e^{-rt'}\frac{v_\omega(\theta')}{i(\theta')} \geq e^{-rt'}\frac{v_\omega(\theta)}{i(\theta)} + e^{-rt}\frac{v_\omega(\theta')}{i(\theta')}.$$

. When the above condition holds, the planner always wants to implement any information stock process Z so that higher types stop first. The condition is equivalent to $(e^{-rt} - e^{-rt'})(v_\omega(\theta)/i(\theta) - v_\omega(\theta')/i(\theta')) \geq 0$. To see that this is satisfied, notice that $-i'(\theta)/i(\theta) \in [0, -v'_L(\theta)/v_L(\theta)]$ implies that $v_\omega(\theta)/i(\theta)$ is increasing in θ for both $\omega \in \{H, L\}$.

Now, we can use montonicity and define the planner's problem as finding the information stock policy Z that maximixes

$$\mathbb{E}\left[\int_z^1 e^{-r\tau(s)}(x\hat{v}_H(s) + (1-x)\hat{v}_L(s))ds \Big| x, z; Z \right], \tag{44}$$

where $\hat{v}_\omega(z) = v_\omega(h^{-1'}(z))$. Notice that the problem is equivalent to (4).

All assumptions in Section 2 hold when $-i'(\theta)/i(\theta) \in [0, -v'_L(\theta)/v_L(\theta)]$ because then $\hat{v}_\omega(z)$ is decreasing in z for both $\omega \in \{H, L\}$, and hence the claim in the proposition immediately follows for this case.

Part (i): experts. Suppose $i'(\theta) \geq 0$.

The monotonicity of the stopping profile follows directly from Lemma 2 in the experts environment because the planner can always decide not to use the additional information from the higher type θ in the proof of Lemma 2. Therefore, we can use (44) as the planner's objective.

In the experts environment, $\hat{v}_H(z)$ need not be decreasing and therefore we need to verify that the policy function x^* we get from Proposition 2 is increasing in z and therefore defines a boundary policy. All other parts of the proof of Proposition 2 remain unchanged even when the stopping payoffs are not monotone.

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To show the monotonicity of x^*, we redo the part of the proof of Proposition 2 that shows that $g(x, z) > 0$.

$$
\begin{aligned}
g(x,z) =& x(1-x)\Big[x\Big(\beta'(z)(\beta(z)-1)\hat{v}'_H(z) - ((\beta(z)-1)\beta''(z) - 2(\beta'(z))^2)\hat{v}_H(z)\Big) \\
& + (1-x)\Big(\beta'(z)\beta(z)\hat{v}'_L(z) - (\beta(z)\beta''(z) - 2(\beta'(z))^2)\hat{v}_L(z)\Big)\Big]\Big/ \\
& \Big[\Big(x(\beta(z)-1)^2\hat{v}_H(z) + (1-x)(\beta(z))^2\hat{v}_L(z)\Big)\beta'(z)\Big] \\
=& x(1-x)\Big[x\Big(\beta'(z)(\beta(z)-1)(v'_H(z) + v_H(z)h^{-1''}(z)(h^{-1'}(z))^{-1}) \\
& - ((\beta(z)-1)\beta''(z) - 2(\beta'(z))^2)v_H(z)\Big) \\
& + (1-x)\Big(\beta'(z)\beta(z)(v'_L(z) + v_L(z)h^{-1''}(z)(h^{-1'}(z))^{-1}) \\
& - (\beta(z)\beta''(z) - 2(\beta'(z))^2)v_L(z)\Big)\Big]\Big/ \\
& \Big[\Big(x(\beta(z)-1)^2 v_H(z) + (1-x)(\beta(z))^2 v_L(z)\Big)\beta'(z)\Big],
\end{aligned}
$$

where we use $v_\omega(z)$ for $v_\omega(h^{-1'}(z)))$ and use that $\hat{v}'_\omega(z) = v'_\omega(z)h^{-1'}(z) + v_\omega(z)h^{-1''}(z)$.

The expression is otherwise equivalent to $g(x, q)$ in (9) but with an additional term in the numerator:

$$
x(1-x)\beta'(z)h^{-1''}(z)(h^{-1'}(z))^{-1})\Big[x(z)(\beta(z)-1)v_H(z) + (1-x)\beta(z)v_L(z)\Big].
\tag{45}
$$

First, notice that $\beta' < 0$, $h^{-1'} < 0$, and $h^{-1''} < 0$ where the last part follows from the assumption that we are in the experts environment. The term inside the brackets is negative for all $x < x^E(z)$ and hence (45) is weakly positive at $x^*(z)$. Then, $g(x, z) > 0$ follows from the proof of Proposition 2 for the original model because the additional term only makes it larger.

$\square$

When high types are extremely fanatical, we may run into trouble: because low types are more informative, the social planner may want to use them first for experimentation. If we restrict to monotone allocations over types, the policy function may be non-increasing because the scaled payoffs are non-monotone. Essentially the social planner may want to bunch some agents to stop together because lower types may have a larger social value of stopping.
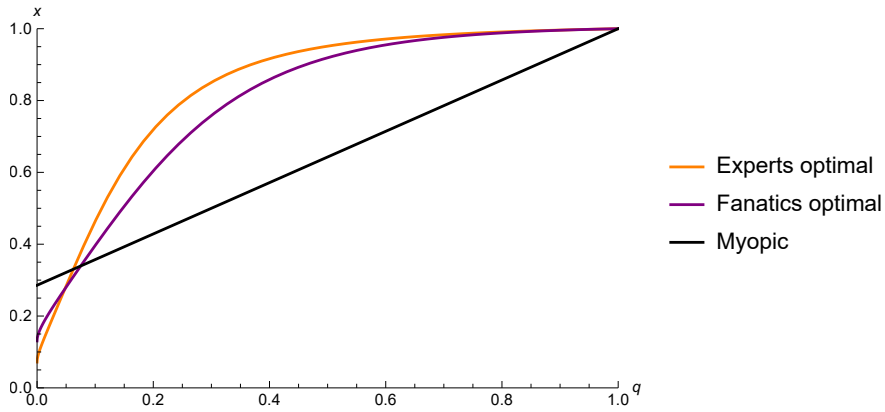
Figure 8: Socially optimal policies for experts (concave $h(q) = 3/2q - 1/2q^2$) and fanatics (convex $h(q) = 1/2q + 1/2q^2$) when $v_H(q) = 1 - q$, $v_L(q) = -0.4 - q$, $\sigma = 0.5$, and $r = 0.1$.

Figure 8 illustrates how heterogeneous informativeness affects the socially optimal policy. As before, the comparison between optimal quantities depends on the informational trade-off between option value and information generation effects. The information generation effect is more pronounced for low q in the expert environment, while it is more pronounced for higher q in the fanatics environment. We see this in Figure 8 as the solution in the expert environment is first below and then quickly rises above the fanatics solution. The expert environment favors relatively early expansions because the high types produce more information than in the fanatics environment.