

Waiting for my neighbors

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Abstract

We introduce a neighborhood structure in a waiting game, where the payoff of stopping increases each time a neighbor stops. We show that the dynamic evolution of the network depends on initial parameters and can take the form of either a shrinking network, where players at the edges stop first, or a fragmenting network where interior players stop first. We find that, in addition to the coordination inefficiency standard in waiting games, the neighborhood structure gives rise to an additional inefficiency linked to the order in which players stop. We discuss an application to technology adoption in networks and analyze how time varying subsidy policies can be used to speed up adoption.

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1 Introduction

There is growing evidence that the decision to adopt a new technology is affected by the decisions of neighbors, i.e. those close either geographically or in terms of social or technological distance (Foster and Rosenzweig 1995, Conley and Udry 2010, Bandiera and Rasul 2006, Atkin et al. 2017). One explanation is that adoption creates spillovers for neighbors that decrease their own adoption costs. These spillovers can be informational or technological. For instance, the initial adopter trains employees or suppliers with this new technology and the mobility of workers or the sharing of suppliers spreads the expertise to connected firms.

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Such environments create incentives for players to wait for their neighbors to adopt. In this paper we introduce a class of problems, *waiting games on networks*, that encompasses the adoption problem described above. War of attrition games have been extensively studied, but the neighborhood structure has to this point been ignored. We show that it leads to a new source of inefficiency in addition to the usual timing inefficiency arising in classical war of attrition games, linked to the order of stopping of players.

To highlight this new source of inefficiency, we use the simplest neighborhood structure which is the line, where each player has either one or two neighbors. Specifically, players are organized on line segments of random length and play an infinite horizon timing game. Each player has to decide when to take an action, we call “stop”. The benefit of the action for an individual at date t depends on the neighbors’ past actions. Whenever a player stops, she increases the payoff of stopping of all her neighbors. This creates incentives for all players to wait in the hope that their neighbor(s) stop before them, i.e. gives rise to the structure of a waiting game.

Each link between two consecutive players is i.i.d drawn at date 0. The probability distribution of the network structure is common knowledge, but players do not observe the realization of the network structure but only their direct neighbors (as in Jackson and Yariv 2005, 2007 or Galeotti et. al. 2010).¹ This implies that a player does not know which of two possible types a given neighbor is: either the neighbor has one neighbor (i.e. she is at the end of the line), or two neighbors (i.e. she is inside the line). We restrict ourselves to symmetric strategies and show that at any point in the game, players share the same belief about the type of an arbitrary neighbor. The endogenous evolution of these beliefs is the key aspect of our analysis as we explain below.

Generically, in a symmetric equilibrium of our game, at any given date, only one type of player mixes between stopping and waiting, while players of the other type strictly prefer to wait. Two very different dynamic evolutions of the network can emerge based on parameters of the model, and in particular on the payoffs of the different types. First, what we call shrinking networks, where players of type 1 (extremities of the line)² initially have more incentives to stop and hence the network shrinks over time. Second, fragmenting networks where players of type 2 (inside the line) initially have more incentives to stop, which leads to a fragmentation of the network in smaller networks over time.

¹For our applications this assumption captures the idea that players are not aware of the full structure of the network.

²The convention is to define type k as a type with k neighbors.

Consider first shrinking networks. Initially, players of type 1 are mixing. As time passes and the unique neighbor of a player has not stopped, the belief about the type of the neighbor evolves. Two countervailing forces affect this belief. First, there is the classic updating of beliefs: since players of type 1 are more likely to stop, as time passes, the player becomes more confident that the neighbor is of type 2. However, there is a second countervailing effect, purely linked to the dynamic evolution of the network structure. Even if the neighbor started off as a type 2, her other neighbor might have stopped in the meantime, making it possible that she has turned into a type 1. We show that these two effects perfectly balance each other in the line network, so that the beliefs that the player is of type 1 stays constant throughout the game and only players at the extremity of the line mix and do so at a constant rate, as if they were playing a classic war of attrition with a single player of a given type.

When players of type 2 initially have more incentives to stop, we have the case of fragmenting networks, where players inside the line stop first and fragment the line into smaller segments. In the case of fragmenting networks, both effects affecting beliefs mentioned above go in the same direction. As time passes and a neighbor has not stopped, players become more confident that she is of type 1. In addition, even if she started as a type 2, her own neighbor might have stopped, changing her into a type 1. Thus, as time passes, the belief that the neighbor is of type 2 decreases. Over time the network fragments into smaller networks. At some date, all players of type 2 will have stopped and only isolated pairs will remain. These pairs will then play a classical war of attrition.

Waiting games give rise to a timing inefficiency: players inefficiently delay stopping in order to benefit from the action of others. We analyze how the timing inefficiency can be alleviated by different subsidy policies. We first assume that the regulator has no access to detailed network information and must use a uniform subsidy that is paid to any player at the moment of stopping. We show that the most effective uniform subsidy is one that declines steeply over time: letting the subsidy payment be lower tomorrow than today gives a strong incentive to stop early. We also show that the network properties matter: in a more connected network, subsidies are more likely to increase welfare. Finally, we go beyond uniform subsidies and demonstrate that if the regulator can base subsidy payments on the neighbors' adoption decisions welfare can be further enhanced.

A key message of the paper is that the network structure also gives rise to another source of inefficiency that we call *order inefficiency*, referring to the order in which players stop. Players, when they decide whether to stop, do not take into account the

positive externality they provide to their neighbors. This implies that, in equilibrium, players inside the line, i.e. with more neighbors, have insufficient private incentives to stop. There is thus a region of parameters where the equilibrium results in a shrinking network whereas the first best would be a fragmenting network. We also show that even in the case of the fragmenting network, the order of exit of players inside the line is important for total welfare. We distinguish *regular fragmenting* (where every other player inside the line exits) from *random fragmenting* where any player inside the line is equally likely to stop.

There are many relevant applications of this model and we highlight two more specifically. The first classical application of war of attrition games is to industry shakeouts. Papers in this literature (Fudenberg and Tirole 1986, Ghemawat and Nalebuff 1985, 1990) ignore the neighborhood structure. In environments where the neighborhood structure matters (where a link represents technological or geographical proximity), for instance the micro brewery sector described in Tremblay et. al. (2005), that experienced periods of rapid entry and subsequent shakeout, our paper highlights that an inefficiency might occur due to the order of exits in the shakeout phase. The second important application, as mentioned at the start of the introduction, is to technology adoption. In this case, subsidies have often been introduced to solve the timing inefficiencies, as discussed in the seminal paper by Stoneman and David (1986) who examine subsidies as instruments to speed up adoption. We show in the last part of the paper that the network structure adds to the effectiveness of such policies. Since every adoption decision entails a positive externality on *all* subsequent adopters, the positive welfare effect of the subsidy propagates through the network.

As Jackson and Zenou (2014) point out, the literature on strategic dynamic games on networks is still limited, and in particular there are no infinite horizon strategic timing games on networks. Most interest has in fact focused on repeated games (Raub and Weesie 1990, Ali and Miller 2013 among others). The core of the mechanism is that punishment of deviations by one neighbor will also impact the payoff of the other neighbors and contagion of bad behavior can thus occur.

Leduc et. al. (2017) consider a related problem, but focus on information diffusion in a two period model. Players need to take an action whose payoff depends on a binary state. If a player takes the action in period 1, all her neighbors learn the state, and the player obtains a referral payoff. The authors solve for the mean-field approximation of the game, as in much of the literature on network diffusion (such as in Jackson and Yariv 2007), and show that agents with low degree have incentives to take the action

early while those with higher degrees free ride.³ We are interested in the fully dynamic game where payoffs depend on the number of neighbors and we solve directly for the perfect Bayesian equilibrium. This allows us to introduce learning about the network structure as the game evolves.

The literature on war of attrition games has been applied to many cases, including the original work on biological competition (Maynard Smith 1974), labor strikes (Kennan and Wilson 1989), industrial organization (Fudenberg and Tirole 1986). Bulow and Klemperer (1999) consider a generalization of the classic model to the case of $n+k$ players competing for n prizes. None of the papers consider the influence of the neighborhood structure, which is the focus of the current study.

2 Model

There is a countable set of players labeled by $i \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$. Each pair of consecutive players i and $i + 1$ are initially connected to each other with probability $\chi \in (0, 1)$, independently of all other consecutive pairs. Hence, the players are organized in a countable set of finite segments, where the length of each segment follows a geometric distribution. We say that two players are neighbors if they are linked to each other. Each player can be of type $k \in \{0, 1, 2\}$, where k denotes the number of her neighbors. We can express the fraction of the players that are initially of type k in terms of χ as:

$$\begin{aligned} q_0 &= (1 - \chi)^2, \\ q_1 &= 2\chi(1 - \chi), \\ q_2 &= \chi^2. \end{aligned} \tag{1}$$

Players only have to decide when to take an action that we call “stop”. Time is continuous and the benefit for an individual stopping at date t depends on how many neighbors she has at that date. If a player stops at time t , and has k neighbors at that date, her realized payoff is

$$\Pi(k, t) = e^{-rt} B_k,$$

where r is the rate of discounting and B_k is the time invariant benefit of stopping for a player with k neighbors. We are interested in a class of games where B_k is a decreasing sequence ($B_2 < B_1 < B_0$). We present foundations for this payoff structure in Section

³There is also a literature on learning from neighbors’ actions, where players take actions repeatedly but without forward looking strategic behavior, see e.g. Bala and Goyal (1998) and Gale and Kariv (2003).

4.

The structure of the neighborhood evolves dynamically. As soon as a player stops, she exits the game. We represent this as a deletion of all her links. Consider a player with initially k neighbors, so that initially her payoff if she decided to stop would be B_k . If one of her neighbors stops, she is left with $k - 1$ neighbors, and her payoff of stopping increases to B_{k-1} . This creates incentives for all players to wait for others to stop.

2.1 Strategies and information

Each player observes her own neighbors, shares a common prior on the network structure but has incomplete information about the realized structure (as in Galeotti et. al. 2010). Since the players only observe the behavior of their neighbors, a private history at t for a player consists of stopping dates of her neighbors up to time t .

It is immediately clear that for a player of type $k = 0$ who has no neighbors, it is strictly dominant to stop immediately. For notational simplicity we ignore type $k = 0$ in the definition of strategies.

Consider the strategy for player i of type $k = 1$ who initially has one neighbor. A pure strategy for such a player is simply a stopping time i.e. $T_1^i \in [0, \infty)$. This stopping time is conditional on her neighbor still remaining in the game: if the only neighbor stops at time $t < T_1^i$, i becomes type $k = 0$ and stops also at time t .

A player of type $k = 2$ who initially has two neighbors has two components in her strategy. First, she must choose when to stop conditional on none of her two neighbors having stopped, i.e. choose $T_2^i \in [0, \infty)$. If one of the two neighbors stops before T_2^i , she becomes type $k = 1$ and must choose when to stop conditioning on the time at which she became type $k = 1$. Therefore, the other component of a pure strategy is a mapping $T_{21}^i(\tau) : [0, \infty) \rightarrow [0, \infty)$ that defines the time to stop when one of the two neighbors stops at time τ . This mapping must satisfy $T_{21}^i(\tau) \geq \tau$ for all $\tau \geq 0$. The entire pure strategy for player i can hence be written as

$$T^i = (T_1^i, (T_2^i, T_{21}^i(\cdot))),$$

and a corresponding behavioral strategy is

$$\sigma^i = \left(\sigma_1^i, \left(\sigma_2^i, \{ \sigma_{21}^i(\tau) \}_{\tau \geq 0} \right) \right),$$

where σ_1^i and σ_2^i are probability distributions over $[0, \infty)$ and $\sigma_{21}^i(\tau)$ is a probability distribution over $[\tau, \infty)$. This definition of strategies implies that two neighbors of a

player are treated symmetrically.

Given a profile σ of behavioral strategies, the outcome of the game consists of realized payoffs and stopping dates as described below.

Take any finite segment of players, and proceed through the following steps:

Step 1: For each player i in the line segment, draw the planned stopping date t^i from the appropriate component of the behavioral strategy (from component σ_1^i if i has one neighbor, or from component σ_2^i if i has two neighbors). Denote the minimum of these stopping times across the players by t , and let each player who chose that time stop at t and get payoff $e^{-rt}B_k$. If, as a result, some of the remaining players becomes type $k = 0$, let those players also stop at t and get payoff $e^{-rt}B_0$. Proceed to step 2.

Step $n \geq 2$: Amongst those players that remain after step $n - 1$, take all such players i that were originally type $k = 2$ but became type $k = 1$ and draw for them a new planned stopping time t^i from $\sigma_{21}^i(t)$. For all the other remaining players keep their planned stopping time unchanged. Then, repeat the same procedure as in step 1, i.e. take again the minimum of the planned stopping times amongst the remaining players, remove those players who chose that stopping time and give them appropriate payoffs, remove all players that became type $k = 0$. If there are players left, go to step $n + 1$.

Since the number of players is finite, all the players will have stopped after a finite number of steps at well defined dates and obtained their payoffs.

2.2 Beliefs

The probability distribution of the network structure is common knowledge to the players, but each player only observes her own neighbors. Each player therefore forms an initial belief on the type of her neighbor(s), either $k = 1$ or $k = 2$. Since a link exists between any two consecutive players independently with probability χ , the initial belief of an arbitrary player about the type of an arbitrary neighbor is:

$$\begin{aligned} p_1 &= 1 - \chi, \\ p_2 &= 1 - p_1 = \chi. \end{aligned}$$

Since there is one-to-one correspondence between parameter χ and initial beliefs p_1 and p_2 , we eliminate χ from the rest of the analysis and track only the players' beliefs about their neighbors' types. Using (1), we can also express the initial beliefs as functions

of the initial fraction of different types of players as:

$$\begin{aligned} p_1 &= \frac{q_1}{q_1 + 2q_2}, \\ p_2 &= \frac{2q_2}{q_1 + 2q_2}. \end{aligned}$$

There are two key properties of this initial belief structure, $P1$ and $P2$, that are important for our analysis.

Property $P1$ (Anonymity): A player $i \in \mathbb{Z}$ has the same belief on the type of each of her neighbors (if she has any), and this belief is the same for all i , regardless of her own type.

Property $P2$ (Independence): If a player has two neighbors, she believes that their types are independently distributed.

As time passes, the players update their beliefs on their neighbors' types based on the equilibrium strategies. We establish in the following Lemma that the two properties $P1$ and $P2$ continue to hold for any date $t > 0$ as long as players use symmetric strategies.

Lemma 1 *With symmetric strategy profiles, $P1$ and $P2$ remain satisfied at all dates $t \geq 0$.*

This result allows us to summarize the belief structure at time t by a vector $p(t) = (p_1(t), p_2(t))$, where $p_2(t) = 1 - p_1(t)$.⁴

2.3 Equilibrium

Our solution concept is Perfect Bayesian Equilibrium. Throughout the paper we treat the players anonymously, and therefore concentrate on symmetric strategy profiles. A strategy profile σ is a symmetric Perfect Bayesian Equilibrium if it is symmetric and (i) the belief $p(t)$ about a neighbor's type is derived from σ via Bayes' rule for all private histories and (ii) the strategy σ is optimal for any private history given the belief $p(t)$ and given that all the other players use the strategy σ .

Although an arbitrary strategy profile is a complex object, a symmetric equilibrium profile can be summarized by the distribution of stopping dates that it induces for any given neighbor of i conditional on i never stopping. By symmetry and Lemma 1, this

⁴Note that even if the belief structure can be expressed as this simple object, it implies a belief on the shape of the entire network. In particular, each player believes that the length of the half-line starting from a neighbor is geometrically distributed with parameter $p_2(t)$.

distribution is the same for all neighbors of i and for all i . Let this distribution be denoted by F . In the following Lemma we summarize a key property of F that holds in any symmetric equilibrium:

Lemma 2 *In a symmetric equilibrium, the stopping date of an arbitrary neighbor has an atomless probability distribution F with full support on $[0, \infty)$.*

Lemma 2 allows us to define for all $t \geq 0$ the hazard rate of stopping of an arbitrary neighbor as

$$\gamma(t) = \frac{F'(t)}{1 - F(t)}.$$

In general, the same hazard rate γ can sometimes admit multiple type breakdowns. In order to reduce this multiplicity, we restrict attention to **Markovian** equilibria such that players who are type 1 at date t and who started off as types 2 play the same continuation strategy from date t regardless of the date at which their neighbor stopped, and also the same continuation strategy from date t as a player who started off as a type 1 from date 0.

In a Markovian equilibrium, the hazard rate γ can be uniquely broken down by type. For each $k \in \{1, 2\}$ such that $p_k(t) > 0$, there exists a function $\lambda_k(t)$ that gives the hazard rate of stopping of a player of type k at date t .⁵ Moreover,

$$\gamma(t) = p_1(t) \lambda_1(t) + p_2(t) \lambda_2(t).$$

In what follows, we will characterize Markovian symmetric equilibria of the model by directly analyzing the properties of the stopping hazard rates λ_k for each type $k = 1, 2$. We show in Appendix C that our main results extend to non Markovian strategies.

3 Waiting for my neighbors: equilibrium characterization

In our model, the heterogeneity between players is due to their position in the line, specifically the number of neighbors that they have. To understand the specific role of the neighborhood structure, we first study a waiting game with heterogenous types, where the source of heterogeneity is not linked to a particular neighborhood structure.

⁵For types k such that $p_k(t) = 0$, the function $\lambda_k(t)$ is indeterminate.

3.1 Benchmark with no neighborhood structure

We consider a game between two players who can have one of two possible types: type 1 gets benefit B_1 if she stops first and B_0 if she stops after the other player and type 2 gets benefit B_2 if first and B_1 if second ($B_0 > B_1 > B_2$). Both players know their type and share a common prior that the other player is of type $j \in \{1, 2\}$ with probability p_j . Consistent with our model with neighbors, the belief about the other player's type is independent of one's own type. We derive the symmetric equilibrium of this game. The shape of the equilibrium depends on the comparison between μ_1 and μ_2 where $\mu_j = \frac{rB_j}{B_{j-1}-B_j}$.

Proposition 1 *If $\mu_j > \mu_k$ (either $j = 1$ and $k = 2$ or the reverse), there exists a date t_j^b such that in all symmetric equilibria:*

- *For $t < t_j^b$ only players of type j mix between the actions stop and wait. Both players expect the other to stop at a rate μ_j .*
- *The posterior belief that the other player is of type j , $p_j(t)$, is decreasing and such that $p_j(t_j^b) = 0$.*
- *For $t \geq t_j^b$ players of type k mix at constant rate μ_k .*

As shown in Proposition 1, in a symmetric equilibrium, only one single type mixes at any point in time. Indeed, when a player of a given type $l \in \{1, 2\}$ is mixing, she needs to be indifferent between the cost of waiting, equal to rB_l and the expected gain if the other player stops, equal to $(B_{l-1} - B_l)$ that accrues with probability μ , where μ denotes the rate of stopping of the other player. Since types are not correlated, μ is independent of the own type and generically only one type can satisfy the indifference condition

$$\mu(B_l - B_{l-1}) = rB_l. \quad (2)$$

Proposition 1 then characterizes the timing of actions. Consider the case where $\mu_1 \equiv \frac{rB_1}{B_0-B_1} > \mu_2 \equiv \frac{rB_2}{B_1-B_2}$. Players of type 1 have more incentives to stop and initially are the only types to mix. The equilibrium mixing rate, as can be seen in equation (2), has to be such that all players share the belief that the other player will stop at rate $\mu = \mu_1$. Note that μ_1 is both a function of the belief that the other player is of type 1 and of the mixing rate λ_1 of players of type 1. We have specifically $\mu_1 = p_1(t)\lambda_1(t)$. As time passes and the other player has not stopped, the posterior $p_1(t)$ that she is of type 1 decreases. At some date t_1^b all types 1 will have stopped. If the two players are still

active, they are then certain that the other is of type 2. Players of type 2 then start mixing at a constant rate μ_2 as in a classical war of attrition.

3.2 Waiting for my neighbors

We now explicitly introduce the neighborhood structure and the heterogeneity between players is then due to the position in the line. Types differ in the number of neighbors they have and thus in terms of payoffs when stopping. The payoffs when stopping are the same as for the benchmark studied above: type 1 who makes benefit B_1 if she stops first and B_0 if she stops after the other player and type 2 who gets benefit B_2 if first and B_1 if second.

There are two key differences with the benchmark model. First, for types 2, the fact of having two neighbors doubles the chances of at least one of them stopping and thus affects the strategic choices. Second, and more important, the types evolve dynamically: if the neighbor of a given player is initially a type 2 but her other neighbor stops, she becomes a type 1. This change in the type of the neighbor is not observed by the player, but the possibility of such a dynamic evolution affects the beliefs about the neighbor's type.

It is important to distinguish two cases depending on the respective values of

$$\bar{\gamma}_1 := \frac{rB_1}{B_0 - B_1} \quad \text{and} \quad \bar{\gamma}_2 := \frac{rB_2}{2(B_1 - B_2)}.$$

We show that the case $\bar{\gamma}_1 > \bar{\gamma}_2$ is one where the players of type 1 mix first.⁶ This gives rise to what we call “shrinking networks” since only the players at the extremities of a line mix and over time the line gets shorter. On the contrary, in the case $\bar{\gamma}_2 > \bar{\gamma}_1$, players of type 2 have initially more incentives to mix. This gives rise to what we call “fragmenting networks”. The initial line is cut at some date into two smaller segments and this process repeats itself over time.

Recall that in the benchmark model of section 3.1, two cases were distinguished based on the respective value of μ_1 and μ_2 , which determined which type was mixing first (here we have $\bar{\gamma}_1 = \mu_1$ but $\bar{\gamma}_2$ is different from μ_2 since it integrates the fact that a type 2 has two neighbors in our current setup). However, both cases are qualitatively equivalent in the benchmark while in the case with a network structure, the two cases are radically different, due to the dynamic evolution of the network structure.

⁶In what follows we will use the notation γ for the expected hazard rate of stopping of an arbitrary player. By extension we use the notation $\bar{\gamma}_1$ and $\bar{\gamma}_2$ here.

3.3 Shrinking networks

We start by considering the case $\bar{\gamma}_1 > \bar{\gamma}_2$. As in the benchmark model, only one type of player can be mixing at any point in time. In this case, players of type 1 have more incentives to mix and hence stop first.

However, the key difference with the benchmark case is that, as players of type 1 are mixing, two forces affect beliefs, as reflected in the following dynamic equation:

$$\dot{p}_1(t) = \underbrace{-\lambda_1(t) p_1(t) (1 - p_1(t))}_{\text{updating beliefs about initial type}} + \underbrace{\gamma(t) p_2(t)}_{\text{probability that type 2 becomes 1}} \quad (3)$$

where, as defined in Section 2, $\lambda_1(t)$ is the hazard rate of stopping of a neighbor of type 1 and $\gamma(t)$ is the expected hazard rate of stopping of an arbitrary neighbor (so that $\gamma(t) = p_1(t) \lambda_1(t)$).

The evolution of beliefs described in (3) reflects the balance between two effects. First, players update their beliefs about their neighbor's types based on the fact they do not see her stopping. Second, the types of neighbors may evolve dynamically since even if the neighbor initially had two neighbors (probability $p_2(t)$), her other neighbor might have stopped in the time interval (probability $\gamma(t)$), thus changing her type into a type 1. The two effects on beliefs go in opposite direction. The first effect makes the player less confident that the neighbor started off as a type 1 but the second makes it more likely that she became one over time. The following result examines the balance between these effects in equilibrium:

Proposition 2 *If $\bar{\gamma}_1 > \bar{\gamma}_2$, a Markovian symmetric equilibrium has the following properties:*

1. *The belief that a random neighbor is of type 1 remains constant, equal to $p_1(0)$ throughout the game and type 1 players mix at constant rate $\lambda_1 = \frac{\bar{\gamma}_1}{p_1(0)}$.*
2. *The expected time before an average member of the network stops is given by $E[T] = (q_1 + q_2) \frac{1}{2\bar{\gamma}_1}$ and is increasing in B_0 , decreasing in B_1 and independent of B_2 .*

Proposition 2 shows that in the Markovian symmetric equilibrium⁷ the two effects on beliefs perfectly balance each other in the case of the line. As a consequence, the

⁷These properties are also satisfied in non Markovian symmetric equilibria, one example of which we provide in the Appendix C.

belief that the neighbor is a type 1 stays constant throughout the game and players play an infinite war of attrition as if they were facing a single player mixing at rate $\bar{\gamma}_1$. Only the players of type 1, by definition positioned at the extremities of the line, mix at any point in time. Overall, the line shrinks in size over time, hence the terminology. Furthermore, as reflected in result 2, since only type 1 players stop during the entire game, decreasing their incentives to stop by increasing B_0 or decreasing B_1 delays stopping. Similarly, since type 1 incentives are independent of B_2 , the equilibrium stopping rate is independent of B_2 .⁸

3.4 Fragmenting networks

We now consider the case $\bar{\gamma}_2 > \bar{\gamma}_1$. We show that in this case, players of type 2 initially have the strongest incentives to stop. As in the previous case the evolution of beliefs about the neighbor's type is the result of two effects: updating based on the fact that the neighbor did not stop and dynamic evolution of beliefs. However in this case both effects go in the same direction and as time passes it becomes increasingly likely that the neighbor is of type 1. Formally, the evolution of the beliefs is given by:

$$\dot{p}_2(t) = \underbrace{-\lambda_2(t)p_2(t)(1-p_2(t))}_{\text{updating beliefs about initial type}} - \underbrace{\gamma(t)p_2(t)}_{\text{probability that type 2 becomes 1}} \quad (4)$$

where $\lambda_2(t)$ is the hazard rate of stopping of a neighbor of type 2 and $\gamma(t)$ is the expected hazard rate of stopping of an arbitrary neighbor (so that $\gamma(t) = p_2(t)\lambda_2(t)$).

As in the benchmark case of section 3.1, $p_2(t)$ is decreasing and at some date t_2 players are sure that their neighbor is not of type 2, i.e. $p_2(t_2) = 0$. At that date, types 1 mix exactly as in the benchmark case.

The rate of stopping by types 2 does not however follow the same dynamics as in the benchmark case. If a type 2 player decides to stop, she gets B_2 as in the benchmark case. When she waits, it is in the hope that one of her two neighbors stops in the meantime, at which point she will become a type 1 with value $V_1(t)$ that varies over time (while it was constant in the benchmark). Thus the stopping rate of a random neighbor is given by:

$$\gamma(t) = \frac{rB_2}{2(V_1(t) - B_2)},$$

⁸In the extreme case where $B_0 = B_1 > B_2$, players stop one by one at date 0, and there is no delay. Indeed, a type 1 player has nothing to wait for in that case, and it is a dominant strategy for her to stop at date 0, as soon as she turns into a type 1.

where the value $V_1(t)$ is defined by the following Bellman equation:

$$V_1(t) = \gamma(t) B_0 dt + (1 - \gamma(t) dt) (1 - r dt) \left(V_1(t) + \dot{V}_1(t) dt \right)$$

Indeed, the payoff of a player of type $k = 1$ at a date t where only types $k = 2$ are mixing is composed of the expected payoff in the period dt , which is B_0 if the neighbor stops (probability $\gamma(t)$), plus the continuation value. As long as players types $k = 1$ strictly prefer to wait, we have $V_1(t) > B_1$, but $V_1(t)$ is strictly decreasing in time.

Proposition 3 *If $\bar{\gamma}_2 > \bar{\gamma}_1$, there exists a date t_2 such that a (Markovian) symmetric equilibrium satisfies:*

- *For $t < t_2$ only types 2 are mixing and the expected rate of stopping of a random neighbor is $\gamma(t) = \frac{rB_2}{2(V_1(t) - B_2)}$, where $V_1(t)$ is the value function of type 1. We have $B_0 > V_1(t) > B_1$ and $\dot{V}_1(t) = -\frac{rB_2(B_0 - V_1(t))}{2(V_1(t) - B_2)} + rV_1(t) < 0$.*
- *At time $t = t_2$, $V_1(t_2) = B_1$ and $p_2(t_2) = 0$.*
- *For $t > t_2$ players of type $k = 1$ mix at a constant hazard rate $\bar{\gamma}_1$.*

Furthermore, if $p_2(0) < \frac{1}{2}$, types 2 are active for a longer period of time than in the benchmark case ($t_2 > t_2^b$).

Compared to the benchmark model, there are two main forces that affect the time t where the players are sure the other player is not of type 2 (i.e. t_2 in the case under consideration and t_2^b in the benchmark model). First, types 2 mix at a lower rate for two reasons: they have two neighbors, so the chance of at least one stopping is higher than in the benchmark model. Furthermore, the value obtained if one neighbor stops, V_1 , is higher than in the benchmark, B_1 . Both these effects imply that there are more incentives to wait and the stopping rate will be lower. At the same time, as time passes, some neighbors of type 2 become type 1 thus decreasing the incentives to wait. If the proportion of types 2 is initially small as indicated in the last result of Proposition 3, the first effect dominates.

The dynamic evolution is very different than in the case of the shrinking network. Only types 2, situated at the heart of the network as opposed to its extremities, initially mix. At some point one of them randomly stops. The initial network is then fragmented in two smaller networks and the same process repeats itself.

3.5 Order inefficiency

We established above that, in equilibrium, the network behaves as a shrinking network rather than a fragmenting network if and only if

$$\begin{aligned}\bar{\gamma}_1 \geq \bar{\gamma}_2 &\Leftrightarrow \frac{B_1}{B_0 - B_1} \geq \frac{B_2}{2(B_1 - B_2)} \\ &\Leftrightarrow 2B_1 \geq B_2 + \frac{B_2}{B_1} B_0 .\end{aligned}\tag{5}$$

We show now that this order of stopping can be socially inefficient. When a player decides to stop, she provides a positive externality to her neighbors as their own payoffs from stopping increase. Because players do not internalize this externality, players with more neighbors might have insufficient incentives to stop. This implies that the players might in equilibrium behave as in a shrinking network while the first best would be a fragmenting network, where the players with more neighbors stop first.⁹

Moreover, the order of exit in the fragmenting case also has an impact on welfare. There are different possible stopping orders that would result in different degrees of fragmentation of the network. Consider two different modes of stopping, which we call *regular fragmenting* where exactly every second player exits before their neighbors (for instance even numbered players stop first) and *random fragmenting*, where any interior player of type 2 can be the first to stop. Regular fragmenting is the stopping order that maximizes the number of players that get payoff B_2 and minimizes the average distance between players.¹⁰ Random fragmenting is the equilibrium stopping order in the case of fragmenting networks.

We contrast in Proposition 4 the socially optimal stopping order to the equilibrium order, where by socially optimal we mean the order that maximizes the sum of the players' payoffs ignoring any costs of delay. We show that whenever $2B_1 \geq B_2 + B_0$, the social planner wants to minimize the degree of fragmentation, i.e. minimize the number of players that get payoff B_2 . Then the shrinking network is the best while a regular fragmenting network is the worst. Conversely, when the condition is violated, the planner wants to maximize the degree of fragmentation. Then the regular fragmenting network gives the highest welfare while the shrinking network is the worst. In both cases the random fragmenting network gives an intermediate level of total welfare. We summarize this discussion in the proposition below.

⁹Note that the timing inefficiency is also the result of ignored externalities.

¹⁰Obtaining regular fragmenting requires information on the exact network details available to the planner but not to the players.

Proposition 4 *Depending on the payoff parameters B_0, B_1, B_2 :*

- *If $2B_1 \geq B_2 + B_0$, in equilibrium the network behaves as a shrinking network, which generates the socially optimal order of stopping.*
- *If $2B_1 \in (B_2 + \frac{B_2}{B_1}B_0, B_2 + B_0)$, in equilibrium the network behaves as a shrinking network while the socially optimal order of stopping is regular fragmenting.*
- *If $2B_1 < B_2 + \frac{B_2}{B_1}B_0$, in equilibrium the network is characterized by random fragmenting while the socially optimal order of stopping is regular fragmenting.*

To illustrate the results, consider the case where the initial network is randomly drawn to be a line of size 5. If players behave as in the shrinking network case, the resulting aggregate payoff is $4B_1 + B_0$. If players behave as in the regular fragmenting networks, where the initial player to stop is constrained to be either player 2 or player 4, the resulting aggregate payoff is $2B_2 + 3B_0$. Player 2 by stopping first, increases the payoff of player 1 from B_1 to B_0 , at a cost of $B_2 - B_1$. Under the condition $2B_1 < B_2 + B_0$, this increases welfare but requires player 2 to internalize this externality. In the case of the random fragmenting network, with 1 chance out of 3, player 3 exits first, leaving players 1 and 2 and players 4 and 5 in a shrinking network. In that sense, the random fragmenting network decreases the degree of fragmentation. If $2B_1 < B_2 + B_0$, this decreases welfare as shown in Proposition 4.¹¹

4 Timing inefficiency and technology adoption in networks

The timing inefficiency, standard in waiting games, is also present when we introduce the neighborhood structure. We describe it here in the context of the adoption of new technologies by firms organized in a line. The action “stop” represents here adopt the technology. When a firm adopts, it decreases its neighbors’ cost of adoption through either technological spillovers or informational spillovers. In this context, we study the effect of subsidy programs aimed at speeding up adoption. Many countries have in place large scale subsidy programs to support adoption of technologies. While a variety of reasons can justify such subsidy programs, we highlight in this paper the role of subsidies to correct the inefficiencies linked to coordination failures.¹²

¹¹Two extreme cases are of interest. In the case where $B_0 = B_1 > B_2$, the first case of Proposition 4 holds. In the case where $B_2 = B_1 < B_0$, the third case of Proposition 4 holds.

¹²Of course, different motivations drive public intervention in these different areas. The main justification for subsidies in the case of environmentally friendly technologies, and to some extent health related

4.1 Application of the model to adoption of technologies

We first examine how our model applies to adoption of technologies and argue that shrinking networks are more relevant in this case.

Consider a situation where spillovers between neighbors are technological, so that a link represents technological proximity between two members of the network.¹³ Upon adoption, the adopting firm trains employees and potentially trains suppliers if the new technology affects the interactions with suppliers. We know that there is large mobility of skilled labor across firms in the same technological areas and that firms situated close to each other often share suppliers. Both effects imply that adoption by one firm may reduce the adoption costs of its neighbors (Jaffe et. al. 1993, Almeida and Kogut 1999).

In this application, a player is characterized by two state variables: a , the number of active neighbors (those who have not yet adopted) and na , the number of inactive neighbors (those who adopted in the past and provided spillovers). The number of inactive neighbors determines the payoff when stopping while the number of active neighbors impacts the incentives to wait. However, in our model, we allow for a single state variable k , the number of neighbors. The results we obtained directly apply to the context of technology adoption as described above if we assume that all players start out with the same number of neighbors, i.e. $a + na = 2$. In this case, keeping track of the number of active neighbors is sufficient, since $na = 2 - a$. We show in Appendix B3 that the equilibrium structure that we identify in Section 3 is preserved if we do not impose the restriction $a + na = 2$ and consider the general case with two state variables. The main difference lies in the transitions from one state to another when a neighbor adopts, i.e. moves from being active to inactive.

Consider as an illustration the following specification of payoffs. Suppose the time invariant benefit of adopting the technology is given by B and denote c_a the cost of adoption for a player who does not benefit from spillovers. The adoption by one neighbor reduces the cost by a factor σ_1 , and the next adoption reduces the cost further by another factor σ_2 . The benefit of adoption for a player with na inactive neighbors (who have already exited) is thus given by $B - (\prod_{l=1}^{na} \sigma_l) c_a$. In the case of 2 neighbors, this implies $B_2 = B - c_a$, $B_1 = B - \sigma_1 c_a$ and $B_0 = B - \sigma_1 \sigma_2 c_a$, since having no active neighbors directly implies that there are 2 inactive neighbors who already provided spillovers.

products, is the internalization of an externality. For agricultural techniques, as reported in Dufflo et al. (2011), there is much less consensus on the source of market failure justifying state intervention. Some cite informational problems while others invoke behavioral biases.

¹³Informational spillovers, due for instance to the fact that firms can observe the adoption techniques used by their neighbors, are formalized in Appendix B1.

Section 3 established that in equilibrium, the network behaves as a shrinking network if and only if $\bar{\gamma}_1 > \bar{\gamma}_2$. Using the parametrization of profits introduced above, we conclude that the condition is satisfied under fairly general conditions on σ_l . This is the case for instance if σ_l is constant, since then $\bar{\gamma}_1 = \frac{r(B-\sigma)c_a}{\sigma(1-\sigma)c_a} > \bar{\gamma}_2 = \frac{r(B-c_a)}{2(1-\sigma)c_a}$. To be in the case of the fragmenting network requires σ_1 to be very small relative to σ_2 , which appears unlikely.¹⁴ The shrinking network case also seems to be the most relevant case empirically: speed of technology diffusion is often described using measures of distance covered by year (see the survey by Geroski 2000).

We thus restrict attention in this section to shrinking networks and we study the effect of subsidy programs aimed at speeding up adoption. To guarantee that the equilibrium is always a shrinking network, we assume throughout this section that B_2 is sufficiently low relative to B_0 and B_1 so that players of type 2 never want to adopt before type 1 players.¹⁵

4.2 Uniform subsidies

We start with the assumption that the regulator cannot identify the types of individual players and hence offers a uniform subsidy payment to any player that adopts at time t . We set the stage by considering the simplest policy, that we call permanent subsidy policy, where a fixed subsidy $s > 0$ is given to a player at the moment of adoption. We will then consider a more general class of time varying policies, where the regulator commits to a path $\mathbf{s} := \{s(t)\}_{t=0}^{\infty}$, where payment $s(t)$ is given to a player that adopts at time t . Finally, we ask how the possibility of random expiration of a policy affects welfare.

Typically there is a deadweight loss of funds raised to finance such subsidy programs. To calculate overall welfare we thus assume throughout the analysis of the different subsidy programs that, for a given subsidy s awarded, the social cost is given by $(1+\alpha)s$, where parameter $\alpha > 0$ measures the welfare loss associated with raising and transferring funds. We will assume throughout the analysis that any subsidy payment must be positive.

To define the social welfare of a subsidy policy, let us consider a player i arbitrarily picked from the population of players, and denote by $B(i)$ and $t(i)$ the random payoff and the time of adoption of that player induced by the policy. The social welfare induced

¹⁴Could be the case if spillovers come from suppliers and a sufficient mass of firms needs to adopt to give incentives for the supplier to also invest in the new technology.

¹⁵A sufficient condition is that $B_2 < \frac{3}{2}B_1 - \frac{1}{2}B_0$. Note that as long as this condition holds, none of the results depend in any way on the exact value of B_2 .

by a subsidy policy \mathbf{s} is

$$W(\mathbf{s}) = \mathbb{E}e^{-rt(i)} [B(i) - \alpha s(t(i))], \quad (6)$$

where the expectation is taken over i , $B(i)$ and $t(i)$.

4.2.1 Deterministic subsidy policies

We start by studying the benchmark case of a permanent subsidy set at some $s > 0$. From the players' perspective this just amounts to replacing payoff terms B_k with $B_k + s$, $k = 0, 1, 2$. We showed in section 3.3 that for shrinking networks, the belief that the neighbor is of type 1, the mixing rate of a type 1 player and the expected stopping rate of a random neighbor remain constant throughout the game. In particular the hazard rate of adoption by an arbitrary neighbor is now given by:

$$\gamma^{pe}(s) = p_1(t) \lambda_1^{pe}(s) = \frac{r(B_1 + s)}{B_0 - B_1}. \quad (7)$$

This is linearly increasing in s so we see that the subsidy speeds up adoption as intended. The downside is that each adoption induces a social loss of αs at the time of adoption.

Denote by $W^{pe}(s)$ the social welfare of the permanent subsidy s . We compute (6) in the Appendix for the permanent subsidy and obtain:

$$\begin{aligned} W^{pe}(s) &= q_0 B_0 + \frac{2(B_1 + s)}{B_0 + B_1 + 2s} \left(\frac{q_1}{2} B_0 + \left(\frac{q_1}{2} + q_2 \right) B_1 \right) \\ &\quad - \alpha s \left(q_0 + \frac{2(B_1 + s)}{B_0 + B_1 + 2s} (q_1 + q_2) \right). \end{aligned} \quad (8)$$

The expression can be interpreted as a decomposition of (6). The first line in the expression gives $\mathbb{E}e^{-rt(i)} B(i)$. The term $q_0 B_0$ corresponds to types $k = 0$ who adopt immediately at $t = 0$. The term $\left(\frac{q_1}{2} B_0 + \left(\frac{q_1}{2} + q_2 \right) B_1 \right)$ gives the expected payoff for the rest of the players. Since in each line segment there are initially two players of type $k = 1$ and only the last player to adopt obtains payoff B_0 , the fraction of players that get B_0 is $q_1/2$. The rest get payoff B_1 . The multiplier $\frac{2(B_1 + s)}{B_0 + B_1 + 2s}$ is the expectation of $e^{-rt(i)}$ conditional on a player not being type $k = 0$ initially. This term is increasing in s and captures the benefit of the subsidy. The second line in the expression gives $\mathbb{E}e^{-rt(i)} \alpha s$, and has a similar interpretation. This term captures the cost of the subsidy policy and is increasing in s .

We show in the appendix that $W^{pe}(s)$ is strictly concave and $(W^{pe})'(0) > 0$ for α

low enough. Since $W^{pe}(0)$ gives the welfare without any subsidy, this implies that for sufficiently low α there exists an optimal permanent subsidy level that increases welfare.

Allowing the subsidy to vary in time can increase welfare. The key intuition is that letting the subsidy payment be lower tomorrow than today increases the opportunity cost of delay and hence increases incentives to adopt immediately. Based on this logic, we focus our attention to policies where the subsidy payment decreases smoothly over time. Formally, assume that the subsidy policy s is continuous, differentiable almost everywhere, and $\dot{s}(t) < 0$ whenever $s(t) > 0$.¹⁶

We look for an equilibrium, where type $k = 1$ is indifferent at all times between adopting and waiting, and adopts with a time varying hazard rate $\lambda_1^{sm}(t)$. As before, p_1 and p_2 remain constant at their initial values $p_1(0)$ and $p_2(0)$ throughout, and the resulting hazard rate of adoption by an arbitrary neighbor can be written as

$$\gamma^{sm}(t) = p_1(0) \lambda_1^{sm}(t).$$

The indifference condition for type 1 between adopting at t and $t + dt$ is:

$$B_1 + s(t) = \gamma^{sm}(t) dt (B_0 + s(t)) + (1 - \gamma^{sm}(t) dt) (1 - rdt) (B_1 + s(t) + \dot{s}(t) dt).$$

Solving this for $\gamma^{sm}(t)$ gives:

$$\gamma^{sm}(t) = \frac{r(B_1 + s(t)) - \dot{s}(t)}{B_0 - B_1}. \quad (9)$$

We see from this equation that both a positive current level of subsidy ($s(t) > 0$) and a negative rate of change in the subsidy path ($\dot{s}(t) < 0$) increase the rate of adoption. Hence, the decline in the subsidy path has two counteracting welfare effects. On one hand, $\dot{s}(t) < 0$ boosts the current rate of adoption (current $\gamma^{sm}(t)$ is higher). On the other hand, $\dot{s}(t) < 0$ reduces the future rate of adoption (future $s(t)$ and thereby future $\gamma^{sm}(t)$ is lower). We analyze this trade-off in the Appendix, and show that the former effect dominates: it is best to let the subsidy decline as fast as possible.

We show that there exists a well defined upper bound for welfare for any subsidy path that starts from a given level s , and we denote this upper bound by $W_\infty^{sm}(s)$. We show that this upper bound is approximated by a policy that declines from s to zero

¹⁶It can be shown that the regulator can never benefit from letting the subsidy increase over time. Neither can the regulator benefit from using a discontinuous subsidy path. The assumption that $s(t)$ is strictly decreasing is made only to simplify the dynamic programming argument that we use in the Appendix to compute the welfare of a policy. Of course, welfare of a weakly decreasing policy can be approximated arbitrarily closely with some strictly decreasing policy.

arbitrarily steeply. The key to understand this result is to note that even if in the limit the policy is in place only for a negligibly short time, the players who according to (9) respond by a hazard rate that scales linearly in the rate of decline will adopt with a non-negligible probability before s hits zero. We show that the limiting probability of “immediate adoption” is given by $1 - e^{-\frac{2s}{B_0 - B_1}}$.¹⁷ Hence, even if the policy operates a very short time, it manages to induce a non-negligible fraction of players to adopt immediately. Of course, in reality it is impossible to induce an arbitrarily steep decline and it is important to understand how intermediate cases perform. We show that any declining subsidy that starts from s will induce a higher social welfare than a permanent subsidy set at s .

We derive in the Appendix an explicit formula for $W_\infty^{sm}(s)$ and note that it is strictly concave with $\lim_{s \rightarrow \infty} W_\infty^{sm}(s) = -\infty$. Hence, there is a unique maximizer $s^* \geq 0$ for $W_\infty^{sm}(s)$. The maximal value W^* is an upper bound for the welfare that any uniform subsidy policy can achieve:

$$W^* := \max_{s \geq 0} W_\infty^{sm}(s).$$

We summarize the discussion in the following proposition:

Proposition 5 *There exist thresholds $0 < \alpha^{pe} < \alpha^*$ such that:*

1. *If $\alpha < \alpha^{pe}$, permanent subsidies increase welfare and there exists some optimal permanent subsidy level $s > 0$. If $\alpha \geq \alpha^{pe}$, no permanent subsidy improves welfare.*
2. *If $\alpha < \alpha^*$, declining subsidy policies increase welfare. If $\alpha \geq \alpha^*$, no declining subsidy improves welfare.*
3. *For any $s > 0$, any declining subsidy with initial level s achieves higher welfare than the permanent subsidy set at s .*
4. *There exists a tight upper bound W^* for declining subsidy policies. No policy obtains exactly W^* , but a subsidy that declines steeply from some $s^* > 0$ to zero gets arbitrarily close.*

Finally we examine how the properties of the network influence the effectiveness of subsidy policies. Recall that we have denoted by χ the probability that players i and $i+1$ are initially connected. This parameter is a measure of how connected the network is. We show in the next proposition that connectedness increases the potential for declining subsidies to increase welfare.

¹⁷This is the probability with which a constant hazard rate $\xi \cdot \frac{2s}{B_0 - B_1}$ produces an arrival before time $1/\xi$ for any $\xi > 0$ and hence also in the limit $\xi \rightarrow \infty$.

Proposition 6 *The more connected the network, the more likely it is that there exists a declining subsidy policy that increases welfare: α^* is increasing in χ .*

4.2.2 Randomly expiring subsidy

So far we have considered deterministic subsidy policies. In reality there are often significant uncertainties about how long these subsidies will be in effect. This could be due to political turnover or even economic shocks that can force the interruption of subsidy programs. Such uncertainty will reduce the value of waiting, and hence may potentially increase the incentives to adopt quickly.

To illustrate the effect of random expiration as clearly as possible, we consider the simplest case, where a permanent subsidy expires at a constant hazard rate. We show that this leads to a welfare improvement over a deterministic permanent subsidy, but cannot match the social value of a deterministically declining subsidy policy.

Suppose that a subsidy is set at level $s > 0$, but expires at some random time that is exponentially distributed with parameter κ . Any player that adopts before expiration gets payoff $B_k + s$, while a player that adopts after expiration gets B_k . We derive an equilibrium, where type $k = 1$ stops at a constant rate $\lambda_1^{ra}(s, \kappa)$ until expiration (after expiration, the game continues as the original game without any subsidies). This adoption rate induces an arbitrary neighbor to adopt at rate:

$$\gamma^{ra}(s, \kappa) = p_1(0) \lambda_1^{ra}(s, \kappa).$$

To derive the equilibrium value of $\gamma^{ra}(s, \kappa)$, note that type $k = 1$ must be indifferent between stopping at t and $t + dt$:

$$\begin{aligned} B_1 + s &= \gamma^{ra}(s, \kappa) dt (1 - rdt) (B_0 + s) + \kappa dt (1 - rdt) B_1 \\ &+ (1 - \gamma^{ra}(s, \kappa) dt) (1 - \kappa dt) (1 - rdt) (B_1 + s). \end{aligned}$$

The equilibrium stopping rate of an arbitrary neighbor is thus given by:

$$\gamma^{ra}(s, \kappa) = \frac{r(B_1 + s) + \kappa s}{B_0 - B_1}. \quad (10)$$

We see from this equation that increasing κ induces a higher rate of adoption.

We compute in the Appendix the total welfare of this policy, denoted $W(s; \kappa)$, and show that it is strictly increasing in κ . This implies in particular that the highest welfare, denoted $W_\infty^{ra}(s)$, is obtained in the limit $\kappa \rightarrow \infty$ and the lowest level is obtained when

$\kappa \rightarrow 0$, where the welfare converges to the level obtained with the permanent subsidy. We also show that $W(s; \kappa)$ is strictly concave in s so that for a fixed κ there is a uniquely optimal level of subsidy.

Notably, both a randomly expiring policy and a smoothly declining policy yield the highest social welfare in the limit, where the policy is in effect only for a vanishingly short time. This stems from the fact that in both cases magnifying the rate of decline increases linearly the hazard rate of adoption, as seen in (9) and (10), which implies that in the limit there is a non-negligible fraction of players that adopt “immediately”, i.e. before the policy is removed. However, this fraction is different in the two cases. Denote by Φ^{ra} and Φ^{sm} the limit (expected) fraction of players that stop before the policy expires in case of randomly expiring and smoothly declining policies, respectively. We show in the Appendix that:

$$\begin{aligned}\Phi^{ra} &= q_0 + (q_1 + q_2) \frac{1}{1 + \frac{B_0 - B_1}{2s}} \\ &< \\ \Phi^{sm} &= q_0 + (q_1 + q_2) \left(1 - e^{-\frac{2s}{B_0 - B_1}}\right).\end{aligned}$$

There is another difference between the two limiting policies. In the case of randomly expiring policy, every player that adopts “immediately” pays s , while in the smoothly declining policy every player that adopts “immediately” pays on average a fraction of s (we compute this fraction explicitly in the Appendix and show that it is somewhat more than half of initial s). Both of these differences favor the smoothly declining policy, and we conclude that $W_\infty^{sm}(s) > W_\infty^{ra}(s)$. We summarize our discussion in the proposition below:

Proposition 7 *A subsidy set at level $s > 0$ that expires at rate $\kappa \geq 0$, induces total welfare $W(s; \kappa)$ with the following properties:*

- $W(s; \kappa) > W^{pe}(s)$ for all $\kappa > 0$,
- $W(s; \kappa)$ is strictly increasing in κ with $W(s; 0) = W^{pe}(s)$ and $\lim_{\kappa \rightarrow \infty} W(s; \kappa) := W_\infty^{ra}(s) < W_\infty^{sm}(s)$.

Figure 1 illustrates the welfare comparison between different uniform subsidy policies. The three uniform subsidy policies (permanent, smoothly declining, and randomly expiring) are drawn as a function of initial subsidy level s , where the randomly expiring policy is in the limit $\kappa \rightarrow \infty$, and the smoothly declining policy is in the limit, where

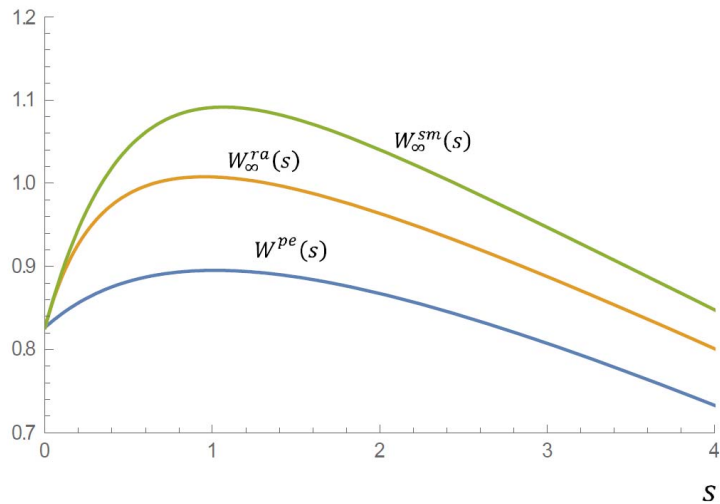


Figure 1: Comparing subsidy programs. Parameter values are: $B_0 = 2$, $B_1 = 1$, $\alpha = 0.1$, $q_0 = 0.04$, $q_1 = 0.32$, $q_2 = 0.64$

the decline is infinitely fast. Note that the optimal level of s is different in each policy. The upper bound for any policy is W^* , which is the maximum value of $W_\infty^{sm}(s)$.

One can of course ask if a more general subsidy policy combining random expiration with a smoothly declining policy could achieve more than any deterministic declining policy. We analyze such policies in Appendix D and show that the answer is negative: the total welfare remains bounded from above by W^* . Random expiration serves as a substitute for declining subsidies to increase the current stopping rate, but does so less efficiently for the two reasons highlighted above.

4.3 Reward when a neighbor adopts

So far we have considered uniform subsidy policies, where a given subsidy is paid to any player that adopts at a given time. The reason for focusing on such policies is the implicit assumption that the regulator does not have access to detailed neighborhood information. In some situations it may however be possible, for example by relying on the agents' self reporting, to base subsidy payments on the neighbors' adoption decisions. This may in effect allow the regulator to target subsidies to certain type of players only.

To demonstrate the potential of such policies, we analyze a neighbor reward policy,

where a fixed reward $m > 0$ is given to any player at the time the first of her neighbors adopts after she has adopted herself. The key observation is that by using such a policy the regulator may target a subsidy payment to type $k = 1$ only. This has two benefits over a corresponding uniform policy: 1) the subsidy payment need not be paid to types $k = 0$ who adopt immediately anyway, 2) a type $k = 1$ has an increased incentive to adopt quickly since she misses the subsidy payment if her neighbor stops before her.

When $B_0 - B_1 \leq m \leq B_0 - B_2$, there is an equilibrium where agents of type 1 adopt immediately, triggering an instantaneous cascade of adoptions. All levels of reward within this range are dominated by $\bar{m} = B_0 - B_1$, which achieves the same adoption pattern than the higher levels at a lower cost. This level of reward achieves the total welfare

$$W^{na}(\bar{m}) = B_0 - (1 + \alpha) \left(q_2 + \frac{q_1}{2} \right) (B_0 - B_1).$$

For a reward $m \leq \bar{m}$, we look for an equilibrium as in the shrinking networks game, where the belief p_1 that the neighbor is of type 1, the mixing rate $\lambda_1^{na}(m)$ of a type 1 player and the expected stopping rate of a random neighbor $\gamma^{na}(m) = p_1(0) \lambda_1^{na}(m)$ remain constant throughout the game. In that case, the policy is equivalent, from the point of view of the agents, to replacing payoff terms B_1 with

$$B_1 + (1 - p_1(0)) \frac{\lambda_1^{na} + \gamma^{na}}{\lambda_1^{na} + \gamma^{na} + r} m + p_1(0) m.$$

In particular the hazard rate of adoption by an arbitrary neighbor must then satisfy

$$\begin{aligned} \gamma^{na} &= \frac{r \left(B_1 + (1 - p_1(0)) \frac{\lambda_1^{na} + \gamma^{na}}{\lambda_1^{na} + \gamma^{na} + r} m + p_1(0) m \right)}{B_0 - \left(B_1 + (1 - p_1(0)) \frac{\lambda_1^{na} + \gamma^{na}}{\lambda_1^{na} + \gamma^{na} + r} m + p_1(0) m \right)} \\ &= \frac{r \left(B_1 + \frac{\gamma_1^{na}(1+p_1(0))+r(p_1(0))^2}{\gamma_1^{na}(1+p_1(0))+rp_1(0)} m \right)}{B_0 - \left(B_1 + \frac{\gamma_1^{na}(1+p_1(0))+r(p_1(0))^2}{\gamma_1^{na}(1+p_1(0))+rp_1(0)} m \right)}. \end{aligned}$$

Rearranging this expression, we see that for any $m \in [0, \bar{m}]$, the equilibrium adoption rate of a neighbor γ^{na} is the unique positive solution of the equation

$$\begin{aligned} &\gamma^{na} \left((B_0 - B_1) (\gamma_1^{na} (1 + p_1(0)) + r p_1(0)) - \left(\gamma_1^{na} (1 + p_1(0)) + r (p_1(0))^2 \right) m \right) \\ &- r B_1 (\gamma_1^{na} (1 + p_1(0)) + r p_1(0)) - r m \left(\gamma_1^{na} (1 + p_1(0)) + r (p_1(0))^2 \right) \\ &= 0. \end{aligned}$$

The left-hand side is increasing in γ^{na} for $\gamma^{na} > 0$ and decreasing in m , so it follows that the function $\gamma^{na}(m)$ is increasing in m . For $m = 0$, $\gamma^{na}(m) = \frac{rB_1}{B_0 - B_1}$ and as m approaches \bar{m} , $\gamma^{na}(m)$ goes to infinity. From the point of view of both the agents and the planner, the policy m is equivalent in terms of payoff to a discriminating permanent subsidy

$$s^* = \frac{\gamma_1^{na} (1 + p_1(0)) + r (p_1(0))^2}{\gamma_1^{na} (1 + p_1(0)) + r p_1(0)} m$$

that is given on the date of adoption exclusively to adopting type 1 agents, not to type 0 agents. As we show formally in the proof of the following result, such a discriminating subsidy policy can provide adoption incentives to type 1 agents at a lower cost than a non-discriminating subsidy policy. We thus have:

Proposition 8 *For any permanent subsidy $s > 0$, there exists a neighbor reward $m \in [0, \min\{B_0 - B_1, s\}]$ that yields a greater social welfare than the permanent subsidy s .*

For simplicity, we have considered a constant reward m to demonstrate the potential of neighbor reward policies to improve upon a constant uniform subsidy. One could in principle also design neighbor reward policies that improve upon time varying uniform policies. This could be done using a time-varying neighbor reward policy $m(t)$, where a player who adopts at time t will get a reward $m(t)$ at the moment when one of her neighbors adopts. By choosing $m(t)$ appropriately, one could guarantee that the expected payoff of an adopting player of type $k = 1$ exactly matches a given time varying subsidy policy $s(t)$. This would in effect allow the designer to create a declining subsidy policy targeted for types $k = 1$ only.

5 Conclusion

In this paper we have studied a waiting game with a network structure, highlighting the application to the adoption decisions of firms that can benefit from positive spillovers due to adoption of neighbors. We have analyzed how different subsidy policies can be used to mitigate the timing inefficiency in such a context. We have also shown that the neighborhood structure gives rise to an additional inefficiency on top of the timing inefficiency standard in waiting games: an inefficiency in the stopping order. This order inefficiency is due to the fact that players do not internalize the positive externalities they can impose on their neighbors by adopting first.

In order to highlight the special dynamics of stopping and the new source of inefficiency due to the neighborhood structure, we have focused our attention to the simplest

possible network structure, the line. It could be the object of interesting future work to extend our results to larger networks. Our current model clearly cannot capture some aspects of networks that could be relevant. For example, real networks often have cycles. Adding such features in the model will be challenging, since it requires modifying our modeling approach, where every player believes the type of each of her neighbors to be identically and independently distributed. Addressing such issues is left for future work.

REFERENCES

- Ali, N. and D. Miller.** 2013. "Enforcing Cooperation in Networked Societies" *working paper*.
- Almeida, P. and B. Kogut.** 1999. "Localization of knowledge and the mobility of engineers in regional networks." *Management Science*, 45, 905-917.
- Atkin D., A. Chaudhry, S. Chaudry, A. K. Khandelwal and E. Verhoogen.** 2017. "Organizational barriers to technology adoption: evidence from soccer-ball producers in Pakistan " *The Quarterly Journal of Economics*, 132(3), 1101-4.
- Bandiera, O., and I. Rasul.** 2006. "Social Networks and Technology Adoption in Northern Mozambique." *Economic Journal*, 116(514), 869-.
- Bala, V. and S. Goyal.** 1998. "Learning from Neighbours." *Review of Economic Studies*, 65, 595-.
- Bulow, J. and P. Klemperer.** 1999. "The Generalized War of Attrition." *American Economic Review*, 89(1), 175-.
- Conley, T. G., and C. R. Udry.** 2010. "Learning about a New Technology: Pineapple in Ghana." *American Economic Review*, 100(1), 35-69.
- Duflo, E., M. Kremer, and J. Robinson.** 2011. "Nudging Farmers to Use Fertilizer: Theory and Experimental Evidence from Kenya." *American Economic Review*, 101(6), 2350-90.
- Foster, A. D., and M. R. Rosenzweig.** 1995. "Learning by Doing and Learning from Others: Human Capital and Technical Change in Agriculture," *Journal of Political Economy* , 103(6), 1176-1209.
- Fudenberg, D. and J. Tirole.** 1986. "A theory of exit in duopoly," *Econometrica*, 54(4), 943-960.
- Gale, D. and S. Kariv.** 2003. "Bayesian learning in social networks," *Games and Economic Behavior*, 45, 329-346.
- Galeotti A., Goyal S., Jackson M.O., Vega-Redondo F. and L. Yariv.** 2010, "Network games," *Review of Economic Studies* 77, 218-244.
- Ghemawat, P., and B. Nalebuff** 1985, "Exit," *Rand Journal of Economics*, 16(2), 184-
- Ghemawat, P., and B. Nalebuff** 1990 "The Devolution of Declining Industries," *Quarterly Journal of Economics*, 105(1), 167-
- Geroski, P. A.,** 2000, "Models of technology diffusion," *Research Policy*, 29(4-5), 603-625.

- Goyal, S.** 2007. "Connections: An Introduction to the Economics of Networks," *Princeton University Press*.
- Jackson, M.O. and Y. Zenou.** 2014. "Games on networks " *Handbook of Game Theory, Vol. 4, Peyton Young and Shmuel Zamir, eds., Elsevier Science*.
- Jackson, M.O., T. Rodriguez-Barraquer, and X. Tan.** 2012. "Social Capital and Social Quilts: Network Patterns of Favor Exchange," *American Economic Review* 102, 1857- 1897.
- Jackson M.O. and L. Yariv.** 2005. "Diffusion on Social Networks," *Economie Publique* 16, 3-16.
- Jackson M.O. and L. Yariv.** 2007. "The diffusion of behavior and equilibrium structure properties on social networks," *American Economic Review Papers and Proceedings* 97, 92-98.
- Jaffe, A. B., Trajtenberg, M. and R.Henderson.** 1993. "Geographic localization of knowledge spillovers as evidenced by patent citations." *Quarterly Journal of Economics*, 108(3), 577-598.
- Kennan, J. and R. Wilson.** 1989. "Strategic bargaining models and interpretation of strike dates." *Journal of Applied Econometrics*, 4, 87-130.
- Leduc, M. and Jackson, M. and R. Johari.** 2017. "Pricing and Referrals in Diffusion on Networks." *Games and Economic Behavior*, 104, 568-594.
- Maynard Smith, J.** 1974. "The theory of games and the evolution of animal conflicts." *Journal of Theoretical Biology*, 47(1), 209-221.
- Raub, W. and J. Weesie.** 1990. "Reputation and efficiency in social interactions: An example of network effects" *American Journal of Sociology* 96, 626-654.
- Stoneman, P. and P. David.** 1986. "Adoption subsidies vs information provision as instruments of technology policy. " *The Economic Journal* 96, 142-150.
- Tremblay, V., N. Iwaskaki and C. Tremblay.** 2005. "The dynamics of industry concentration for US Micro and Macro brewers " *Review of Industrial Organization* 26, 307-324.
- World Bank.** 2007. "World Development Report 2008: Agriculture For Development."

Appendix A: Proofs

Proof of Lemma 1

As explained in the main text, $P1$ and $P2$ hold at $t = 0$.

We first prove that $P2$ is satisfied at all histories on path. Consider player i and a history at date t , on path for this player, at which i has not stopped. The type at date t of each neighbor j of i is affected only by the actions of the set $N_s^{i,j}$ of agents (excluding i , but including j) that are indirectly connected to i through j at dates $s \in [0, t]$, (and of course by the strategy of player i). For any two distinct neighbors j and j' of i , and for all s , we have $N_s^{i,j} \cap N_s^{i,j'} = \emptyset$. Thus the strategies of agents in $N_t^{i,j}$ are independent across all neighbors j , conditionally on i having not stopped until date s , because the strategies of the players are not correlated. Since player i updates his beliefs according to Bayes' rule, and since player i believes at date 0 that his neighbors' types are independently distributed, it follows that at date t , player i still believes that the types of her neighbors are independently distributed. This proves that $P2$ is satisfied at any history on path for player i .

We now prove that $P1$ is satisfied at all histories on path. Let i and i' be two distinct agents, with neighbors j and j' . For all $s \in [0, t]$, let $G_s^{i,j}$ the set of links of the line that links the agents in $N_s^{i,j}$ and let $G_s^{i',j'}$ be the analogous object for agents in $N_s^{i',j'}$. Because conditions $P1$ and $P2$ hold at date 0, and because the beliefs of the agents at that date are identical, it follows that the belief of player i about the structure of $G_0^{i,j}$ is identical to the belief of player i' about the structure of $G_0^{i',j'}$. That is, these two agents have identical beliefs about the sequence of links, ignoring the labels of the agents in those structures. Then, because $G_t^{i,j}$ (respectively $G_t^{i',j'}$) is only affected by the actions of the players in the line $(N_s^{i,j}, G_s^{i,j})$ (respectively in the line $(N_s^{i',j'}, G_s^{i',j'})$) and that these lines are identically distributed at all date $s < t$ and all player's strategies are independent, it follows that $(N_t^{i,j}, G_t^{i,j})$ and $(N_t^{i',j'}, G_t^{i',j'})$ are identically distributed as well.

Proof of Lemma 2

Consider an arbitrary symmetric equilibrium σ and let F be the distribution of stopping dates of an arbitrary neighbor of an arbitrary player i conditional on i never stopping. Let $T \in \mathbb{R} \cup \{+\infty\}$ be the least upper bound of the support of F . We prove the lemma through the following steps:

1. The cdf F has no atoms. By contradiction, suppose that F has an atom at date t . Then consider a realization of the equilibrium outcome, where player i of type $k \geq 1$

stops exactly at date t . Since F has an atom at t , i expects each of his neighbors to stop at date t with positive probability. It is then a profitable deviation for her not to stop at date t , but rather at date $t + dt$, as the waiting cost of doing so is proportional to dt whereas the expected gain of delaying is bounded from below by $(F(t^+) - F(t^-))(B_{k-1} - B_k) > 0$. A contradiction.

2. There is no interval of positive length included in $[0, T]$ over which F is constant. By contradiction, suppose that \underline{t} and \bar{t} are such that $0 \leq \underline{t} < \bar{t} < +\infty$, and F is constant on $[\underline{t}, \bar{t}]$, but not on a right-neighborhood of \bar{t} . This implies that there is at least one type k such that $\lambda_k(t) > 0$ in a right neighborhood of \bar{t} . Consider a realization of the equilibrium outcome, and suppose that player i of type k with at least one neighbor stops at date $\bar{t} + \varepsilon$, where ε is arbitrarily small. Then stopping instead at date \underline{t} (at which the type of i was the same) would have been a profitable deviation for player i . Evaluated at date 0, the cost of doing such a change is approximately equal to $\gamma(t)(B_{k+1} - B_k)e^{-r(t+\varepsilon)\varepsilon}$, which is small as ε goes to 0. Meanwhile, the benefit of doing so is bounded below by $B_{k+1}e^{-r(\varepsilon+\bar{t}-\underline{t})} > 0$, which exceeds the cost. A contradiction.
3. It must be that $T = +\infty$. To see this, suppose by contradiction that $T < +\infty$. In a realization of the outcome, suppose that an player i stops at date $T - \varepsilon$. Since she knows that all remaining neighbors will exist between $T - \varepsilon$ and T , it is well worth delaying stopping after date T . By doing so, player i strictly gains by stopping ε periods later, the cost of which is bounded above by $e^{-r\varepsilon}B_k\varepsilon$ and the benefit of which is bounded below by $e^{-rT}(B_{k-1} - B_k) > 0$. Thus it must be that $T = +\infty$.

Proof of Proposition 1

We prove the result in the case $\mu_1 > \mu_2$. The other case is perfectly symmetric.

In a symmetric equilibrium, each player faces the same probability distribution for the other player's stopping time, which we denote by $F(t)$, as in the main model. By the same arguments as in Lemma 2, the distribution has no atoms, and its support is $[0, \infty)$. We may hence describe the equilibrium by hazard rates of the two types, $\lambda_1(t)$ and $\lambda_2(t)$, where at least one of them is non-zero for each $t \geq 0$. Letting $p_1(t)$ denote the posterior probability at t that the other player is of type 1, we have

$$\frac{f(\tau)}{(1 - F(\tau))} = p_1(t)\lambda_1(t) + (1 - p_1(t))\lambda_2(t).$$

The expected payoff for type j of the strategy “stop at time τ if the other player has not yet stopped” is given by:

$$W_j(\tau) = \left[\int_0^\tau e^{-rt} (B_{j-1}) f(t) dt + (1 - F(\tau)) e^{-r\tau} B_j \right].$$

Differentiating this, we have

$$\frac{dW_j(\tau)}{d\tau} = e^{-r\tau} B_{j-1} f(\tau) - f(\tau) e^{-r\tau} B_j - r(1 - F(\tau)) e^{-r\tau} B_j,$$

and it follows that

$$\frac{dW_j(\tau)}{d\tau} > (=) (<) 0 \iff \frac{f(\tau)}{(1 - F(\tau))} > (=) (<) r \frac{B_j}{B_{j-1} - B_j} \equiv \mu_j.$$

For a player of type j to mix in an interval $[t, t']$, he must be indifferent between stopping at any date $\tau \in [t, t']$, and we must have:

$$\frac{dW_j(\tau)}{d\tau} = 0 \iff \frac{f(\tau)}{(1 - F(\tau))} = \mu_j.$$

Since $\mu_1 \neq \mu_2$, only one type can be mixing in an interval. Suppose that type j mixes in some interval $[0, t']$ and the other type $k \neq j$ is willing to delay. Then we must have

$$\frac{dW_k(\tau)}{d\tau} \geq 0 \iff \frac{f(\tau)}{(1 - F(\tau))} \geq \mu_k$$

for $\tau \in [0, t']$. Since $\mu_1 > \mu_2$, it must be that the type mixing initially is type $j = 1$ and the type that is waiting is $k = 2$, and so

$$\frac{f(\tau)}{(1 - F(\tau))} = p_1(t) \lambda_1(t) = \mu_1.$$

The updated belief that the other player is of type 1 is then given by Bayes' rule:

$$p_1(t + dt) = \frac{p_1(t)(1 - \lambda_1(t)dt)}{p_1(t)(1 - \lambda_1(t)dt) + (1 - p_1(t))}$$

So that

$$\frac{p_1(t + dt) - p_1(t)}{dt} = \frac{1}{dt} \frac{p_1(t)(1 - \lambda_1(t)dt) - p_1(t)(p_1(t)(1 - \lambda_1(t)dt) + (1 - p_1(t)))}{p_1(t)(1 - \lambda_1(t)dt) + (1 - p_1(t))}$$

Taking limits we have:

$$\dot{p}_1(t) = -\lambda_1(t)p_1(t)(1-p_1(t)).$$

Since $\mu_1 = p_1(t)\lambda_1(t)$, we can reexpress as:

$$\dot{p}_1(t) = -\mu_1(1-p_1(t)).$$

The solution of this differential equation is:

$$1-p_1(t) = (1-p_1(0))e^{-\mu_1 t},$$

and so $p_1(t)$ is strictly decreasing over time. Thus there exists a time t_1^b such that $p_1(t_1^b) = 0$, which we can solve as $t_1^b = -\frac{\ln(1-p_1(0))}{\mu_1}$. After that date only players of type 2 are left, and the continuation game is a standard complete information waiting game with the unique symmetric equilibrium where the players mix at constant rate $\lambda_2(t) = \mu_2$.

Proof of Proposition 2

By the same argument as in the proof of Proposition 1, there must be some initial phase $[0, t']$ during which type 1 randomizes and type 2 waits, and where

$$\gamma(t) = \lambda_1(t)p_1(t) = \bar{\gamma}_1.$$

We next establish the result that as long as type 1 randomizes and type 2 waits, $p_1(t)$ remains constant, and so the initial phase never ends. Suppose that type 1 stops at rate $\lambda_1(t)$ and type 2 waits so that $\lambda_2(t) = 0$. We define two events:

- *NE* (no stopping) the event that no stopping takes place in the interval $[t, t + \epsilon]$.
- *CS* (change state) the event that the neighbor changes state during the interval $[t, t + \epsilon]$, which can only mean that his other neighbor stopped, i.e. he moved from being a type 2 to a type 1.

Using these notations, we have:

$$\begin{aligned} p_1(t + \epsilon) &= \frac{P[k = 2 \cap NE \cap CS]}{P[NE]} + \frac{P[k = 1 \cap NE \cap CS^C]}{P[NE]} \\ &= \frac{p_2(t)(1 - \lambda_1(t)\epsilon)\gamma_1(t)\epsilon}{P[NE]} + \frac{P[NE|k = 1 \cap SC^C]P[k = 1 \cap SC^C]}{P[NE|k = 1]p_1(t) + (1 - p_1(t))} \end{aligned}$$

We now examine:

$$\begin{aligned}
\frac{p_1(t+\epsilon) - p_1(t)}{\epsilon} &= \frac{p_2(t)(1 - \lambda_1(t)\epsilon)\gamma_1(t)}{P[NE]} + \frac{1}{\epsilon} \frac{p_1(t)(1 - \lambda_1(t)\epsilon) - p_1(t)(p_1(t)(1 - \lambda_1(t)\epsilon) + (1 - p_1(t)))}{P[NE]} \\
&= \frac{p_2(t)(1 - \lambda_1(t)\epsilon)\gamma_1(t)}{P[NE]} + \frac{1}{\epsilon} \frac{p_1(t)(1 - p_1(t))\lambda_1(t)\epsilon}{P[NE]} \\
&= \frac{p_2(t)(1 - \lambda_1(t)\epsilon)\gamma_1(t)}{P[NE]} + \frac{p_1(t)(1 - p_1(t))\lambda_1(t)}{P[NE]}.
\end{aligned}$$

Taking the limit when ϵ goes to zero, $P[NE]$ converges to one and so

$$\dot{p}_1(t) = \gamma_1(t)(1 - p_1(t)) - \lambda_1(t)p_1(t)(1 - p_1(t)).$$

Finally, by definition, $\gamma_1(t) = \lambda_1(t)p_1(t)$, so that

$$\dot{p}_1(t) = 0.$$

Given that $p_1(t)$ and $\gamma_1(t) = \bar{\gamma}_1$ do not depend on time, the rate of mixing of types 1, $\lambda_1(t)$, also remains constant and is equal to $\lambda_1 = \frac{\bar{\gamma}_1}{p_1(0)}$. This establishes the first part of the proposition.

We next derive the average time before stopping of a random member of the network. If the player is of type 0 (probability q_0), she stops immediately. If she is of type 1 (probability q_1), her stopping rate is $\lambda_1 + \bar{\gamma}_1$, since she stops either because of her own mixing or because a neighbor stops. Finally, if she is of type 2, she first needs to transition to being a type 1, which occurs at a rate $2\bar{\gamma}_1$, then follows the same dynamic as a type 1. Overall the expected waiting time is given by:

$$\begin{aligned}
E[T] &= q_0 \cdot 0 + q_1 \frac{1}{\lambda_1 + \bar{\gamma}_1} + q_2 \left[\frac{1}{2\bar{\gamma}_1} + \frac{1}{\lambda_1 + \bar{\gamma}_1} \right] \\
&= q_2 \frac{1}{2\bar{\gamma}_1} + (q_1 + q_2) \frac{1}{\lambda_1 + \bar{\gamma}_1}.
\end{aligned}$$

We showed above that $\lambda_1 = \frac{\bar{\gamma}_1}{p_1(0)}$. Furthermore, we showed in Section 2.2 that

$$p_1(0) = \frac{q_1}{q_1 + 2q_2}.$$

Replacing these in the expression for $E[T]$, we get:

$$E[T] = (q_1 + q_2) \frac{1}{2\bar{\gamma}_1}.$$

Being inversely proportional to $\bar{\gamma}_1$, $E[T]$ is increasing in B_0 , decreasing in B_1 and independent of B_2 . Furthermore, since $\bar{\gamma}_1$ is independent of q_1 and q_2 , $E[T]$ is overall increasing in $q_1 + q_2$.

Proof Proposition 3

Following the same arguments as in Proposition 2, we can establish that while players of type 2 are mixing, the beliefs evolve according to:

$$\dot{p}_2(t) = -\gamma(t)p_2(t) - \lambda_2(t)p_2(t)(1 - p_2(t)).$$

Given that $\gamma(t) = \lambda_2(t)p_2(t)$, we obtain that

$$\dot{p}_2(t) = -\gamma(t),$$

i.e.

$$p_2(t) = p_2(0) - \int_0^t \gamma(s) ds.$$

We see that $p_2(t)$ is a strictly decreasing function with derivative bounded away from zero, so there is a date t_2 such that $p_2(t_2) = 0$.

As we argued in the main text, we have

$$\gamma(t) = \frac{rB_2}{2(V_1(t) - B_2)},$$

where the value $V_1(t)$ is defined by the following Bellman equation:

$$V_1(t) = \gamma(t) B_0 dt + (1 - \gamma(t) dt) (1 - r dt) \left(V_1(t) + \dot{V}_1(t) dt \right).$$

Using the value of $\gamma(t)$, we obtain:

$$\dot{V}_1(t) = -\frac{rB_2(B_0 - V_1(t))}{2(V_1(t) - B_2)} + rV_1(t) < 0. \tag{11}$$

To establish the last result, we compare the values of $\dot{p}_2(t)$ in the two cases. Here we have:

$$\dot{p}_2(t) = -\frac{rB_2}{2(V_1(t) - B_2)}.$$

In the benchmark case we had:

$$\dot{p}_2(t) = -(1 - p_2(t)) \frac{rB_2}{(B_1(t) - B_2)}.$$

Given that $V_1(t) \geq B_1$ and $p_2(t) \leq p_2(0) < \frac{1}{2}$, the posterior probability decreases faster in the benchmark case, so that $t_2 > t_2^b$.

Proof of Proposition 4

Take an arbitrary line segment of n players and consider all feasible orders in which those n players can stop. In all the arguments, the players are numbered from 1 to n from left to right. Denote by n_k , $k = 0, 1, 2$, the number of players that get payoff B_k in some stopping order (i.e. number of players who have k neighbors at the date when they stop). Every player eventually stops in each stopping order, so we must have $n_0 + n_1 + n_2 = n$.

Our model allows for the possibility that two neighbors stop simultaneously. However, that is never optimal in terms of total welfare, since by stopping sequentially (even with a negligible lag between the stopping decisions), the payoff of one of the two players jumps up from B_k to B_{k-1} . Therefore, when considering the stopping order that maximizes the total welfare, we ignore the possibility of simultaneous stopping, and take as the set of feasible stopping orders the set of permutations of the players in the segment.

We aim to express the total welfare from an arbitrary stopping order (permutation) as a function of n_2 . As a first step, we consider all the feasible values of n_2 across all possible stopping orders. We claim that

$$\begin{aligned} n_2 &\in \left\{ 0, \dots, \frac{N-1}{2} \right\} \text{ if } n \text{ is odd,} \\ n_2 &\in \left\{ 0, \dots, \frac{N-2}{2} \right\} \text{ if } n \text{ is even.} \end{aligned} \tag{12}$$

To prove this claim, it suffices to note that the smallest possible value $n_2 = 0$ is trivially obtained in the case of a shrinking network, i.e. in a case where the players stop in the order 1, 2, 3, ... (or alternatively $n, n-1, n-2, \dots$). The largest possible value for n_2 is obtained in any sequence where all the even numbered players stop before the odd numbered players (i.e. a regular fragmenting). Such a sequence gives $n_2 = \frac{n-1}{2}$ if n is odd and $n_2 = \frac{n-2}{2}$ if n is even. Any interior value for n_2 is obtained, for example, in a sequence, where the first n_2 even numbered players stop first (each of those players gets B_2), and after that all the remaining players stop in an increasing sequence (and get either B_0 or B_1 each).

As a second step, we claim that in any stopping order, we have

$$n_0 = n_2 + 1. \quad (13)$$

We prove this claim by induction. For $n = 1$ and $n = 2$, the only feasible values of n_0 and n_2 are obviously $n_0 = 1$ and $n_2 = 0$, so (13) holds. Take a line segment of length n where $n = 3, 4, \dots$, and suppose as an induction hypothesis that (13) holds for all line segments of length $k < n$. There are two cases to consider. First, if the first player to stop in that segment is either player 1 or n (i.e. one of the end nodes), then this player gets B_1 and the line shrinks to length $n - 1$. By the induction hypothesis, (13) holds for that reduced line and therefore (13) holds for the original line of length n as well. Second, if the first player to stop is amongst players $2, \dots, n - 1$, she gets B_2 and the line segment splits into two shorter line segments of lengths n' and n'' with $n' + n'' = n - 1$. By the induction hypothesis, we have $n'_0 = n'_2 + 1$ and $n''_0 = n''_2 + 1$, where n'_k and n''_k denote the number of players in the two shorter line segments that get payoff B_k . The total number of players that get B_2 is then $n_2 = 1 + n'_2 + n''_2$ (where 1 is added because the first player to stop in the original line did get B_2) and the total number of players that get B_0 is $n_0 = n'_0 + n''_0$. Combining these equations gives $n_0 = n_2 + 1$ i.e. (13) holds for the original line segment as well.

As a third step, we claim that in any stopping order, we have

$$n_1 = n - 2n_2 - 1. \quad (14)$$

This follows simply from combining (13) and $n = n_0 + n_1 + n_2$, and solving for n_1 .

We can now use (13) and (14) to compute the total welfare in an arbitrary stopping order as a function of n_2 :

$$\begin{aligned} W(n_2) &= n_0 B_0 + n_1 B_1 + n_2 B_2 \\ &= (n_2 + 1) B_0 + (N - 2n_2 - 1) B_1 + n_2 B_2 \\ &= B_0 + (N - 1) B_1 + (B_0 + B_2 - 2B_1) n_2. \end{aligned}$$

We can see from this equation that if $2B_1 < B_0 + B_2$, the total welfare is maximized by choosing the highest possible value of n_2 . By (12) this is obtained with regular fragmenting, where every even numbered player stops first. If $2B_1 > B_0 + B_2$, then the total welfare is maximized by choosing $n_2 = 0$, which is obtained in a shrinking network.

Proof of Proposition 5

We first derive a formula for the social welfare for a given smoothly declining subsidy path $s(t)$. As shown in the main text, the induced stopping rate at time t is

$$\gamma(t) = \frac{r(B_1 + s(t)) - \dot{s}(t)}{B_0 - B_1}. \quad (15)$$

Note that $\gamma(t)$ represents the rate at which an arbitrary *neighbor* of a player stops and this rate is different from the rate of stopping of a randomly picked player, which is what we need for the welfare computations. Let us denote the stopping rate of a randomly picked player by $\tilde{\gamma}(t)$. To compute it, note that for $t > 0$, only types $k = 1$ and $k = 2$ remain (types $k = 0$ stop immediately), and the mix of types remains fixed over time in the case of a shrinking network. Therefore, at any time $t > 0$, the fraction of types $k = 1$ in the population is given by $q_1/(q_1 + q_2)$. A player of type $k = 1$ may stop within dt for two reasons. With probability $\lambda(t) dt$ she stops due to her own randomization and with probability $\gamma(t) dt$ her neighbor stops, which turns her into type $k = 0$ that stops immediately. Noting that types $k = 2$ stop at rate zero in equilibrium, we can express $\tilde{\gamma}(t)$ in terms of $\gamma(t)$:

$$\begin{aligned} \tilde{\gamma}(t) &= \frac{q_1}{q_1 + q_2} (p_1 \lambda(t) + \gamma(t)) \\ &= \frac{q_1}{q_1 + q_2} \left(\frac{q_1 + 2q_2}{q_1} \gamma(t) + \gamma(t) \right) \\ &= 2\gamma(t). \end{aligned}$$

Since the composition of types remains constant, we can hence express the expectation (6) for a policy $\mathbf{s} = \{s(t)\}_{t=0}^{\infty}$ as:

$$W(\mathbf{s}) = q_0(B_0 - \alpha s_0) + \int_0^{\infty} \left(\frac{q_1}{2} B_0 + \left(\frac{q_1}{2} + q_2 \right) B_1 - (q_1 + q_2) \alpha s(t) \right) 2\gamma(t) e^{-\int_0^t 2\gamma(\tau) d\tau} e^{-rt} dt, \quad (16)$$

where $s_0 := s(0)$ and where $\gamma(t)$ is given by (15).

Let us first consider the special case of a permanent subsidy set at level $s \geq 0$. Then we have

$$\gamma(t) = \frac{r(B_1 + s)}{B_0 - B_1}$$

for all $t \geq 0$, and (16) can be computed explicitly:

$$W^{pe}(s) = q_0 B_0 + \frac{2(B_1 + s)}{B_0 + B_1 + 2s} \left(\frac{q_1}{2} B_0 + \left(\frac{q_1}{2} + q_2 \right) B_1 \right) - \alpha s \left(q_0 + \frac{2(B_1 + s)}{B_0 + B_1 + 2s} (q_1 + q_2) \right). \quad (17)$$

It is easy to show by direct computation that $W^{pe}(s)$ is concave in s , and

$$\begin{aligned} (W^{pe})'(0) &> (<) 0 \\ &\iff \\ \alpha &< (>) \alpha^{pe}, \end{aligned}$$

where

$$\alpha^{pe} := \frac{q_0 2B_1^2 + q_1 (B_0^2 + B_1^2) + q_2 2B_0 B_1 - 2B_1^2}{(B_0 + B_1) ((B_0 - B_1) q_0 + 2B_1)} > 0.$$

Since $W^{pe}(0)$ gives the total welfare without any subsidy, this proves result 1 of Proposition 5.

Let us now fix some smoothly declining policy $\mathbf{s} = \{s(t)\}_{t=0}^\infty$. Assume that the policy starts from some positive level $s_0 = s(0)$ and then falls continuously to zero in finite time T so that $s(t) > 0$ and $\dot{s}(t) < 0$ for $t < T$, and $s(t) \equiv 0$ for $t \geq T$. Also, assume that the policy hits zero at a slope bounded away from zero, i.e. $\dot{s}(T) := \lim_{t \rightarrow T} \dot{s}(t) < 0$. Given such a policy \mathbf{s} , we next express (16) as a solution to a differential equation. We first write (16) as

$$W(\mathbf{s}) = q_0 (B - \alpha s_0) + (q_1 + q_2) V(\mathbf{s}),$$

where

$$V(\mathbf{s}) := \int_0^\infty (B - \alpha s(t)) 2 \frac{r(B_1 + s(t)) - \dot{s}(t)}{B_0 - B_1} e^{-\int_0^t 2 \frac{r(B_1 + s(\tau)) - \dot{s}(\tau)}{B_0 - B_1} d\tau} e^{-rt} dt \quad (18)$$

and

$$B := \frac{q_1}{2(q_1 + q_2)} B_0 + \frac{q_1 + 2q_2}{2(q_1 + q_2)} B_1. \quad (19)$$

Noting that $s(t)$ is strictly monotonic, we can express (18) as a function of the current subsidy level s as the state variable. For $s \in [0, s_0]$, let $U(s)$ denote the continuation present value of (18) from the moment $s^{-1}(s)$ where $s(t)$ hits level s (conditional on

the player not having stopped yet):

$$U(s) := \int_{s^{-1}(s)}^{\infty} (B - \alpha s(t)) 2 \frac{r(B_1 + s(t)) - \dot{s}(t)}{B_0 - B_1} e^{-\int_{s^{-1}(s)}^t 2 \frac{r(B_1 + s(t)) - \dot{s}(t)}{B_0 - B_1} dt} e^{-r(t - s^{-1}(s))} dt.$$

Note that with this notation, we have $U(s_0) = V(\mathbf{s})$. We will now derive a differential equation for $U(s)$. Let $\eta(s)$ denote the steepness of the subsidy policy at t for which $s(t) = s$, i.e. $\eta(s) := -\dot{s}(s^{-1}(s))$. Since we have assumed $\dot{s}(t) < 0$ for all $t \in [0, T]$, we have $\eta(s) > 0$ for all $s \in [0, s_0]$. Decomposing $U(s)$ into instant payoff and continuation value dt later, we can write:

$$\begin{aligned} U(s) &= 2 \frac{r(B_1 + s) + \eta(s)}{B_0 - B_1} (B - \alpha s) \cdot dt \\ &\quad + \left(1 - 2 \frac{r(B_1 + s(t)) + \eta(s)}{B_0 - B_1} \cdot dt \right) e^{-r \cdot dt} (U(s) - U'(s) \eta(s) dt). \end{aligned}$$

Rearranging, dividing by dt , and ignoring remaining terms of order dt and higher, this turns into the following differential equation:

$$U'(s) = \frac{2}{B_0 - B_1} \left[-\frac{r}{\eta(s)} \left(\left(\frac{B_0 + B_1}{2} + s \right) U(s) - (B_1 + s)(B - \alpha s) \right) + (B - \alpha s - U(s)) \right]. \quad (20)$$

To solve this, we need a boundary condition, obtained by computing the continuation value after subsidy has declined to zero, i.e. computing (18) for $s(t) \equiv \dot{s}(t) \equiv 0$ for all t :

$$U(0) = \frac{2BB_1}{B_0 + B_1}. \quad (21)$$

The differential equation (20) coupled with initial condition (21) has a unique solution $U(s)$ for $s \in [0, s_0]$. Noting that $V(\mathbf{s}) = U(s_0)$, we can express (16) as

$$W(\mathbf{s}) = q_0 (B - \alpha s_0) + (q_1 + q_2) U(s_0).$$

We show next that the solution through (20) implies item 3 of the proposition. Substitute (19) in (17) to write the value of the permanent subsidy as

$$W^{pe}(s) = q_0 (B_0 - \alpha s) + (1 - q_0) U^{pe}(s),$$

where

$$U^{pe}(s) = \frac{2(B_1 + s)}{B_0 + B_1 + 2s} (B - \alpha s).$$

By inspecting (20), we see that the term $\left(\left(\frac{B_0+B_1}{2} + s\right) U(s) - (B_1 + s)(B - \alpha s)\right)$ multiplied by $-r/\eta(s)$ vanishes for $U(s) = U^{pe}(s)$. For any policy where $\eta(s) < 0$ for all $s > 0$, the solution to (20) gives $U(s) > U^{pe}(s)$. It follows that for any smoothly declining s satisfying our assumptions, we have $W(s) > W^{pe}(s_0)$. Unlike in the main body of the paper, we have assumed so far that there is a bound T such that $s(t) = 0$ for $t \geq T$. Note, however, that we can approximate arbitrarily closely the welfare of any policy that declines to zero only asymptotically by some policy that declines to zero in a finite time, and this means that the conclusion continues to hold for any such policy. Finally, it is very easy to check that the same conclusion holds for a declining policy that is bounded away from zero. To see this, let $\underline{s} := \inf_{t \geq 0} s(t) > 0$. Then we can make a simple change of variables, where B_k is replaced by $B_k + \underline{s}$, $k = 0, 1, 2$, so that the subsidy falls to zero after replacement and the result applies. We can conclude that the total welfare for any declining policy that starts from some $s_0 > 0$ is higher than the welfare for a permanent subsidy set at $s = s_0$. This implies result 3 of the Proposition.

Finally, let us consider the choice over all declining policies. We can see from (20) that there is no optimal solution since for every $s > 0$, $U'(s)$ is maximized by letting $\eta(s) \rightarrow \infty$ (as long as the term $\left(\left(\frac{B_0+B_1}{2} + s\right) U(s) - (B_1 + s)(B - \alpha s)\right)$ is positive, which is the case for any declining subsidy policy). We can, however, analyze the limit $\eta(s) \rightarrow \infty$ to get an upper bound for the value that can be obtained by any declining policy. In this limit, (20) reduces to:

$$U'(s) = \frac{2}{B_0 - B_1} [(B - \alpha s - U(s))].$$

Using again the boundary condition (21), this has a unique solution:

$$\begin{aligned} U_\infty^{sm}(s) \quad : \quad &= \left(1 - e^{-\frac{2s}{B_0-B_1}}\right) B + e^{-\frac{2s}{B_0-B_1}} \frac{2B_1B}{B_0+B_1} \\ &- \left(s - \frac{1}{2}(B_0 - B_1) \left(1 - e^{-\frac{2s}{B_0-B_1}}\right)\right) \alpha. \end{aligned} \quad (22)$$

We now establish that $U_\infty^{sm}(s) > U(s)$, for all $s > 0$, where $U(s)$ is a solution of (20) for some arbitrary positive-value function $\eta(s)$ which also satisfies (21). Consider the difference

$$(U_\infty^{sm}(s) - U(s)) e^{\frac{s(B_0-B_1)}{2}}.$$

The derivative of this function with respect to s equals

$$\begin{aligned}
& (U_\infty^{sm'}(s) - U'(s)) e^{\frac{s(B_0-B_1)}{2}} + \frac{(B_0 - B_1)}{2} (U_\infty^{sm}(s) - U(s)) e^{\frac{s(B_0-B_1)}{2}} \\
&= \left(\frac{r}{\eta(s)} \left(\left(\frac{B_0 + B_1}{2} + s \right) U(s) - (B_1 + s)(B - \alpha s) \right) \right) e^{\frac{s(B_0-B_1)}{2}} \\
&= \frac{1}{\left(\frac{B_0+B_1}{2} + s \right)} \frac{r}{\eta(s)} (U(s) - U^{pe}(s)) e^{\frac{s(B_0-B_1)}{2}} > 0
\end{aligned}$$

for all s , where the last inequality holds because $U(s) > U^{pe}(s)$ for all $s > 0$. Since in addition $U_\infty^{sm}(0) = U(0)$, it follows that $U_\infty^{sm}(s) > U(s)$ for all $s > 0$. Hence this gives us a well defined tight upper bound for the welfare that can be obtained by any declining subsidy path that starts from a given $s(0) > s$.

With some rearranging we can write the limiting total welfare as:

$$\begin{aligned}
W_\infty^{sm}(s) &= q_0(B_0 - \alpha s) + (1 - q_0)U_\infty^{sm}(s) \\
&= q_0 B_0 + \left(\left(1 - e^{-\frac{2s}{B_0-B_1}} \right) + e^{-\frac{2s}{B_0-B_1}} \frac{2B_1}{B_0+B_1} \right) \left(\frac{q_1}{2} B_0 + \left(\frac{q_1}{2} + q_2 \right) B_1 \right) \\
&\quad - \alpha s \left(q_0 + \left(1 - e^{-\frac{2s}{B_0-B_1}} \right) \left(\frac{1}{1 - e^{-\frac{2s}{B_0-B_1}}} - \frac{B_0 - B_1}{2s} \right) (q_1 + q_2) \right), \quad (23)
\end{aligned}$$

which is instructive to interpret as a decomposition of (6). Intuitively, we are analyzing the welfare under a policy that declines "very" fast from s to zero. Even if this decline takes a negligibly short time, the players who respond by a hazard rate that scales linearly in the rate of decline, will adopt with a non-negligible probability before s hits zero. This probability of "immediate adoption" is given by the term $1 - e^{-\frac{2s}{B_0-B_1}}$.¹⁸ The total fraction of players that exit "immediately" is then $q_0 + \left(1 - e^{-\frac{2s}{B_0-B_1}} \right) (q_1 + q_2)$.

It is now straightforward to interpret the first line of (23). The term that requires explanation is $\left(\left(1 - e^{-\frac{2s}{B_0-B_1}} \right) + e^{-\frac{2s}{B_0-B_1}} \frac{2B_1}{B_0+B_1} \right)$, which corresponds to $\mathbb{E}e^{-rt^{(i)}}$ for types $k = 1$ and $k = 2$. This is decomposed into two parts since fraction $\left(1 - e^{-\frac{2s}{B_0-B_1}} \right)$ stop "immediately", and the remaining fraction $e^{-\frac{2s}{B_0-B_1}}$ adopt slowly when the subsidy is no longer in effect, and $\frac{2B_1}{B_0+B_1} < 1$ gives $\mathbb{E}e^{-rt^{(i)}}$ for those players.

The second line of (23) accounts for the financial cost of the policy. Fraction q_0 adopt immediately and induce financial cost αs . The fraction $\left(1 - e^{-\frac{2s}{B_0-B_1}} \right) (q_1 + q_2)$ adopt before s hits zero, but their financial cost is weighted by the term $\left(\frac{1}{1 - e^{-\frac{2s}{B_0-B_1}}} - \frac{B_0 - B_1}{2s} \right)$.

¹⁸This is the probability with which a constant hazard rate $\xi \cdot \frac{2s}{B_0-B_1}$ produces an arrival before time $1/\xi$ for any $\xi > 0$ and hence also in the limit $\xi \rightarrow \infty$.

This always takes a value in $(\frac{1}{2}, 1)$, and it captures the expected share of the initial subsidy level s that a player obtains, conditional on adopting before subsidy becomes zero. This term reflects another benefit to the regulator of using a declining subsidy: the realized subsidy payments become lower as time goes on.

We can show by direct computation that $W_\infty^{sm}(s)$ is strictly concave and for every $s > 0$, we have $W_\infty^{sm}(s) > W^{pe}(s)$. By direct computation from (23) we get:

$$(W^{sm})'(0) = \frac{(1 - q_0) 2B}{B_0 + B_1} - \alpha q_0.$$

We see here that

$$(W^{sm})'(0) > 0 \Leftrightarrow \alpha < \frac{1 - q_0}{q_0} \frac{2B}{B_0 + B_1}.$$

Letting

$$\alpha^* := \frac{1 - q_0}{q_0} \frac{2B}{B_0 + B_1},$$

we have shown that there is an $\alpha^* > \alpha^{pe}$ such that $(W_\infty^{sm})'(0) > 0$ iff $\alpha < \alpha^*$. We then denote

$$W^* = \max_{s \geq 0} W_\infty^{sm}(s).$$

Hence, for all $\alpha < \alpha^*$, we can find a declining subsidy that attains social welfare arbitrarily close to W^* , and so is strictly higher than the welfare without any subsidy.

Proof of Proposition 6

We have to show that

$$\alpha^* := \frac{1 - q_0}{q_0} \frac{2B}{B_0 + B_1}$$

is increasing in χ . Noting that only q_0 and B depend on χ , this amounts to showing that

$$\frac{1 - q_0}{q_0} B$$

is increasing in χ . We have

$$\begin{aligned} q_0 &= (1 - \chi)^2, \\ q_1 &= 2\chi(1 - \chi), \\ q_2 &= \chi^2. \end{aligned}$$

Substituting these in (19) we can write B in terms of χ :

$$B := \frac{1 - \chi}{2 - \chi} B_0 + \frac{1}{2 - \chi} B_1.$$

We now get

$$\frac{d}{d\chi} \left(\frac{1 - q_0}{q_0} B \right) = \frac{B_0 (1 - \chi) + B_1 (1 + \chi)}{(1 - \chi)^3} > 0.$$

Proof of Proposition 7

We can write the expected social welfare of a subsidy set at s and expiring at rate κ as:

$$W^{ra}(s, \kappa) = q_0 (B_0 - \alpha s) + (1 - q_0) V^{ra}(s, \kappa),$$

where $V^{ra}(s, \kappa)$ is the expected social welfare induced by a randomly picked player that is not type $k = 0$. Following the same steps as in the proof of Proposition 5, such a player adopts with rate $2\gamma(s, \kappa)$ as long as the policy is in place, and induces expected social welfare $B - \alpha s$ at the moment of adoption, where

$$B := \frac{q_1}{2(q_1 + q_2)} B_0 + \frac{q_1 + 2q_2}{2(q_1 + q_2)} B_1.$$

If the subsidy expires before that player adopts, the continuation social welfare is given by

$$V^c = \frac{2B_1 B}{B_0 + B_1}.$$

Hence, we can write:

$$\begin{aligned} V^{ra}(s, \kappa) &= \int_0^\infty \kappa e^{-\kappa t} \left[\int_0^t 2\gamma(s, \kappa) e^{-2\gamma(s, \kappa)u} e^{-ru} (B - \alpha s) du + e^{-rt} e^{-2\gamma(s, \kappa)t} V^c \right] dt \\ &= \frac{2\gamma(s, \kappa) (B - \alpha s) + \kappa \frac{2B_1 B}{B_0 + B_1}}{2\gamma(s, \kappa) + \kappa + r} = \frac{2 \frac{r(B_1 + s) + \kappa s}{B_0 - B_1} (B - \alpha s) + \kappa \frac{2B_1 B}{B_0 + B_1}}{2 \frac{r(B_1 + s) + \kappa s}{B_0 - B_1} + \kappa + r}. \end{aligned}$$

V^{ra} is strictly increasing in κ and $W^{pe}(s, 0) = W^{pe}(s)$. It is also easy to show that V^{ra} is strictly concave.

In the limit $\kappa \rightarrow \infty$, we have

$$W_\infty^{ra}(s) = q_0 (B_0 - \alpha s) + (q_1 + q_2) V_\infty^{ra}(s),$$

where

$$\begin{aligned}
V_{\infty}^{ra}(s) & : = \lim_{\kappa \rightarrow \infty} V^{ra}(s, \kappa) \\
& = \frac{1}{1 + \frac{B_0 - B_1}{2s}} (B - \alpha s) \\
& \quad + \left(1 - \frac{1}{1 + \frac{B_0 - B_1}{2s}} \right) \frac{2B_1 B}{B_0 + B_1}.
\end{aligned}$$

To compare this to $W_{\infty}^{sm}(s)$, it is instructive to substitute in B to get:

$$\begin{aligned}
W_{\infty}^{ra}(s) & = q_0 B_0 + \left(\frac{1}{1 + \frac{B_0 - B_1}{2s}} + \frac{\frac{B_0 - B_1}{2s}}{1 + \frac{B_0 - B_1}{2s}} \frac{2B_1}{B_0 + B_1} \right) \left(\frac{q_1}{2} B_0 + \left(\frac{q_1}{2} + q_2 \right) B_1 \right) \\
& \quad - \alpha s \left(q_0 + \frac{1}{1 + \frac{B_0 - B_1}{2s}} (q_1 + q_2) \right). \tag{24}
\end{aligned}$$

It is now straightforward to check that $W_{\infty}^{ra}(s) < W_{\infty}^{sm}(s)$. Intuitively, there are two differences between $W_{\infty}^{ra}(s)$ and $W_{\infty}^{sm}(s)$. First, the term $\frac{1}{1 + \frac{B_0 - B_1}{2s}}$ is the probability of "immediate adoption" in the case of randomly expiring subsidy, and corresponds to the term $1 - e^{-\frac{2s}{B_0 - B_1}}$ in the case of smoothly declining policy. Since

$$\frac{1}{1 + \frac{B_0 - B_1}{2s}} < 1 - e^{-\frac{2s}{B_0 - B_1}},$$

the fraction of players that adopt immediate is lower in the case of randomly expiring subsidy. Second, every player that adopts "immediately" pays the full cost s with the randomly expiring subsidy, in contrast to the average cost

$$\left(\frac{1}{1 - e^{-\frac{2s}{B_0 - B_1}}} - \frac{B_0 - B_1}{2s} \right) s < s$$

in the case of smoothly declining policy.

Proof of Proposition 8

We show that any constant adoption rate γ implemented by a permanent subsidy s handed to any adopting agent is also implemented by a restricted permanent subsidy s^* handed only to adopting type 1 agents, with $s^* < s$. This will then establish that the social planner can implement any given constant neighbor adoption rate γ at lower cost under a neighbor reward policy than under a permanent subsidy.

For any $\gamma \geq \bar{\gamma}_1$, the permanent subsidy s required to implement γ is

$$s = (B_0 - B_1) \left(\frac{\gamma}{r} \right) - B_1 \geq 0$$

whereas the restricted subsidy s^* required to implement the same neighbor adoption rate γ satisfies

$$\begin{aligned} s^* &= \frac{(B_0 - B_1) \left(\frac{\gamma}{r} \right) - B_1}{\frac{\gamma}{r} + 1} \\ &= \frac{s(B_0 - B_1)}{s + B_0} \in [0, s]. \end{aligned}$$

These two policies implement an identical joint distribution of adoption rates and types at the time of adoption. The amount s^* of the restricted permanent subsidy is lower than the amount s of the permanent subsidy. Moreover, the restricted permanent subsidy is paid only to agents who are of type 1 at the time of adoption, unlike the permanent subsidy, which is also paid to type 0. It follows that the restricted permanent subsidy s^* yields a higher social welfare than the permanent subsidy s . Moreover, the restricted subsidy s^* is payoff equivalent to the neighbor reward program m , with

$$m = \left(1 + \frac{p_1(1-p_1)(B_0 - B_1)}{(s + B_1)(1 + p_1) + p_1^2(B_0 - B_1)} \right) s^*.$$

In other words, the permanent subsidy $s > 0$ yields a lower social welfare than the neighbor reward

$$m = \left(1 + \frac{p_1(1-p_1)(B_0 - B_1)}{(s + B_1)(1 + p_1) + p_1^2(B_0 - B_1)} \right) \frac{s}{s + B_0} (B_0 - B_1),$$

which is an amount in $[0, B_0 - B_1]$, since

$$\begin{aligned} \left(1 + \frac{p_1(1-p_1)(B_0 - B_1)}{(s + B_1)(1 + p_1) + p_1^2(B_0 - B_1)} \right) \frac{s}{s + B_0} &\leq \left(1 + \frac{p_1(1-p_1)B_0}{s(1 + p_1) + p_1^2 B_0} \right) \frac{s}{s + B_0} \\ &\leq \left(1 + \frac{\frac{1}{4}B_0}{s} \right) \frac{s}{s + B_0} \\ &= \frac{s + \frac{1}{4}B_0}{s + B_0} \\ &\leq 1. \end{aligned}$$

Appendix B

B1: Informational spillovers

We present a specific model involving informational spillovers across neighbors in the case of the line. Suppose that the cost of adopting depends on the technique used. There are two choices to be made when adopting, for instance different organizational dimensions, $a_1 \in \{L, R\}$ and $a_2 \in \{L, R\}$. The state of nature, described by $\theta = \{\theta_1, \theta_2\}$ determines which adoption technique is less costly. Specifically, the cost of adoption is $c = c_1 + c_2$ where $c_i = c_l 1_{a_i=\theta_i} + c_h 1_{a_i \neq \theta_i}$, i.e. the cost is minimized when the technique used matches the state. When a player observes her neighbor, with probability $1/2$, she learns perfectly about dimension 1 and with probability $1/2$ about dimension 2. What is learned does not depend on the choice the neighbor actually made, which ensures that there is no inference made on the information the neighbor's neighbor held.

In this case

$$\begin{aligned} B_2 &= B - 2\frac{1}{2}(c_l + c_h) = B - (c_l + c_h), \\ B_1 &= B - c_l - \frac{1}{2}(c_l + c_h) = B - \left(\frac{3}{2}c_l + \frac{1}{2}c_h\right), \\ B_0 &= B - c_l - \frac{1}{2}(c_l) - \frac{1}{4}(c_l + c_h) = B - \left(\frac{7}{4}c_l + \frac{1}{4}c_h\right). \end{aligned}$$

So that

$$\begin{aligned} B_0 - B_1 &= \frac{1}{4}(c_h - c_l), \\ B_1 - B_2 &= \frac{1}{2}(c_h - c_l). \end{aligned}$$

In this case we have $\gamma_1 > \gamma_2$, so that this setup will naturally correspond to the shrinking network setup.

B2: War of attrition

We present here a more classical version of the war of attrition, adding as in the rest of the paper the network structure. Firms decide when to exit, where exit is irreversible. Staying in costs $c > 0$ per unit of time, but there is no discounting.

Once both neighbors of a firm exit, the remaining isolated firm gets prize B . As in the rest of the paper, each player only observes whether her neighbors are active or not, but cannot see the status of any other player in the network.

We show there exists a symmetric equilibrium, characterized by a date $t' > 0$ such that within $(0, t')$ all those players who have two active neighbors mix, and within (t', ∞)

there are only players with one active neighbor left (i.e. isolated pairs of players) who play a standard war of attrition with each other.

Denote by $V(t)$ the value of a player, who has one active neighbor left (so that one of her two neighbors have exited). We have $V(t) > 0$ for $t \in (0, t')$ and $V(t') = 0$.

Let us denote by $\gamma(t)$ the hazard rate with which an arbitrary neighbor exits at time t , where $t \in (0, t')$. For a randomizing player to be indifferent, the benefit of delaying exit by dt must equate the cost of doing so, i.e. $2\gamma(t) dtV(t) = cdt$, so that

$$2\gamma(t) V(t) = c,$$

or

$$\gamma(t) = \frac{c}{2V(t)}. \quad (25)$$

The Bellman equation for the player who has only one neighbor left can be written:

$$V(t) = \gamma(t) dtB + (1 - \gamma(t) dt) \left(V(t) + \dot{V}(t) dt \right) - cdt,$$

which gives

$$\dot{V}(t) = -\gamma(t) (B - V(t)) + c. \quad (26)$$

Plugging (25) in (26) gives us a differential equation for $V(t)$:

$$\dot{V}(t) = -\frac{cB}{2V(t)} + \frac{3}{2}c.$$

Starting with any initial value $V(0)$ such that $0 < V(0) < \frac{B}{3}$ this has a solution $V(t)$ that is decreasing and hits zero at some time point t' .

B3: Generalization with two state variables

In the application to the adoption of technologies, a more general model should keep track of two state variables:

- a the number of active neighbors
- na the number of inactive neighbors

Types are thus described by (a, na) where $a \in \{0, 1, 2\}$ and $na \in \{0, 1, 2\}$. A random member of the network can be of types $(2, 0)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, $(0, 1)$ or $(0, 0)$. In the model used in the core of the paper, we restrict ourselves to one state variable. The implicit assumption we make is that $a + na = 2$, i.e. everyone starts with the same

number of neighbors, some active and some inactive. Thus in the main part of the paper there were only three possible types (2, 0), (1, 1), and (0, 2). We now show that the general pattern is preserved with a slight complication due to the existence of types (1, 0). Types (0, 2), (0, 1) or (0, 0) do not have any active neighbors and therefore stop immediately regardless whether they have 0, 1 or 2 inactive neighbors.

As in the main model we introduce some important measures:

$$\begin{aligned}\bar{\gamma}_{(1,0)} &:= \frac{rB_2}{B_1 - B_2}, \\ \bar{\gamma}_{(1,1)} &:= \frac{rB_1}{B_0 - B_1}, \\ \bar{\gamma}_{(2,0)} &:= \frac{rB_2}{2(B_1 - B_2)}.\end{aligned}$$

The equivalence between types here and in the model of section 3 implies that $\bar{\gamma}_{(1,1)} = \bar{\gamma}_1$ and $\bar{\gamma}_{(2,0)} = \bar{\gamma}_2$. We consider two cases: $\bar{\gamma}_{(1,0)} > \bar{\gamma}_{(1,1)}$ and $\bar{\gamma}_{(1,0)} < \bar{\gamma}_{(1,1)}$.

Case 1: $\bar{\gamma}_{(1,0)} > \bar{\gamma}_{(1,1)}$

In this case types (1, 0) have the highest incentives to stop. Indeed these types always have a higher incentive to stop than types (2, 0), since they get the same benefit from stopping B_2 , but they get lower benefit of waiting $\mu(B_1 - B_2)$, whereas types (2, 0) get benefit $(2\mu(V_1 - B_2))$ with $V_1 > B_1$). We now describe the evolution of beliefs.

$$\dot{p}_{(1,0)}(t) = -\lambda(t)p_{(1,0)}(t)(1 - p_{(1,0)}(t)) < 0.$$

As time passes, players become less confident that their neighbor is of type (1, 0). Whereas in section 3 there were two countervailing forces affecting beliefs, here the second force is not present since types (2, 0), if their other neighbor happens to stop, will turn into a type (1, 1), not a type (1, 0).

Thus at some date $t_{(1,0)}$ all types (1, 0) will have stopped. We are then back to the case studied in section 3 with only types (1, 1) and (2, 0). Depending on the relative size of $\bar{\gamma}_{(1,1)} := \frac{rB_1}{B_0 - B_1}$ and $\bar{\gamma}_{(2,0)} := \frac{rB_2}{2(B_1 - B_2)}$, we will be either in the case of shrinking or of fragmenting networks.

Case 2: $\bar{\gamma}_{(1,1)} > \bar{\gamma}_{(1,0)}$

Types (1, 1) initially mix. The evolution of beliefs is given by:

$$\begin{aligned}\dot{p}_{(1,1)}(t) &= -\lambda(t)p_{(1,1)}(t)(1 - p_{(1,1)}(t)) + \bar{\gamma}_{(1,1)}(t)p_{(2,0)}(t) \\ &= -\bar{\gamma}_{(1,1)}(1 - p_{(1,1)}(t) - p_{(2,0)}(t)) < 0\end{aligned}$$

In this case, as in the case studied in section 3, there are two forces affecting the belief $p_{(1,1)}(t)$. However, the dominating effect is the evolution of beliefs and as time passes, active members of the network become less confident that their neighbor is of type (1, 1). At some date $t_{(1,1)}$, among active members of the networks, only types (1, 0) and (2, 0) remain. The networks are therefore formed of lines of random sizes with types (1, 0) at the extremities. Types (1, 0) then have a strictly higher incentive to adopt. As soon as a type (1, 0) adopts, the neighbor, if he is of type (2, 0), transforms into a type (1, 1) and thus immediately adopts. Thus stopping by a type (1, 0) creates an immediate cascade that immediately covers the entire line. It is therefore as if types (1, 0) were playing a waiting game with no type uncertainty. They therefore mix at rate $\bar{\gamma}_{(1,0)}$ and as soon as one adopts, so does the entire line.

Appendix C: Non-Markovian equilibria

Both in the shrinking and fragmenting network cases, there can be non-Markovian Equilibria, where the agents use the realizations of their neighbor's exit dates as randomization devices for their own dates of exit. Importantly, such equilibria are associated with the same distribution $F(t)$ of dates at which a neighbor stops, and also with the same hazard rate $\gamma(t)$ at which a neighbor stops.

Shrinking networks

In the shrinking network case, a simple (and extreme) example of such an equilibrium is the following. In every component, the two players who start off as types 1 from the beginning of the game mix at constant rate $\lambda_1(t) = \bar{\gamma}_1$. Type 2 players never stop, unless one of their neighbors stops, in which case they follow immediately, as soon as they turn into types 1.

Under these strategies, the belief $p_1(t)$ about a neighbor remains constant equal to $p_1(0)$, exactly like in the case of this Markovian equilibrium, but for different reasons. First, a failure to stop is not informative about a neighbor's type, since a type 1 or a type 2 neighbor are equally likely to stop at any given time: a type 1 neighbor, on her own initiative, and a type 2 in reaction to her other neighbor's exit. Thus the belief updating effect is null. Second, a neighbor who was previously a type 2 cannot possibly have turned into a type 1, otherwise she would have stopped immediately. Thus the evolving type effect is also absent. Overall, the belief about a neighbor's type remains constant.

In this non-Markovian equilibrium, the distribution of the stopping date of an average player is exponential with parameter $\frac{2\bar{\gamma}_1}{q_1+q_2q_1+q_2}$, as in the Markovian case. As a result, the average time before an average member of the network stops is the same as in the Markovian equilibrium

$$E[T] = (q_1 + q_2) \frac{1}{2\bar{\gamma}_1}.$$

Note also that the distribution of the stopping date is the same for all agents initially of type either 1 or 2, since all agents of a given component stop at the same date.

This is different in the Markovian equilibrium. There, the distribution is different for players who are initially type 1 and the ones who are initially type 2. The distribution of the stopping date of an agent who is initially of type 1 is also exponential with rate $\frac{2(q_1+q_2)\bar{\gamma}_1}{q_1}$. For agents who are initially type 2, the stopping date is the sum of two variables, each of which follows an exponential distribution, the first with rate $2\bar{\gamma}_1$ and the second with rate $\frac{2(q_1+q_2)\bar{\gamma}_1}{q_1}$. But the distribution of the stopping date of an average player is the same in both the Markovian and non-Markovian equilibrium.

The main observable (and testable) difference between the two equilibria is in the joint distribution of the stopping time profiles. While the non-Markovian equilibrium example has all the agents exiting at the same date, there is some dispersion of exit dates in the Markovian equilibrium.¹⁹

Fragmenting networks

Fragmenting networks too admit non-Markovian equilibria, but their properties differ from the Markovian ones even less than in the shrinking case. In one instance of such an equilibrium, agents who are still active at date t_2 and are of type $k = 1$ at date t_2 could choose a stopping date that is an increasing function ϕ of the date at which their neighbor who stopped prior to t_2 did it. For an appropriately chosen function ϕ , this is an equilibrium.

¹⁹It is easy to construct other non-Markovian equilibria. For example, consider all the convex combinations of the Markovian equilibrium and our non-Markovian example. Or equilibria where players play the Markovian strategy in some set of dates and the non-Markovian example strategy in the complementary set of dates.

Appendix D: General uniform subsidy policy that combines decline and random expiration

We analyze in this Appendix a general uniform subsidy policy $s(t)$ that starts from some s_0 , declines smoothly over time and, in addition, expires randomly at a time-varying hazard rate $\kappa(t)$. We will show below that adding random expiration cannot increase welfare when the rate of decline of $s(t)$ is already fast.

Formally, a general uniform policy can be written as $\mathbf{s} = \{s(t), \kappa(t)\}_{t=0}^{\infty}$, where $s(t)$ is a continuously differentiable function with $s(t) > 0$ and $\dot{s}(t) < 0$ for $t \geq 0$, and where $\kappa(t)$ defines the probability distribution of the date of expiration τ :

$$\Pr(\tau \leq t) = 1 - e^{-\int_0^t \kappa(\tau) d\tau}.$$

The policy gives a lump sum subsidy payment $s(t)$ to any player that stops at $t < \tau$ (i.e. before the policy expires), while a player that stops at some $t \geq \tau$ gets no subsidy.

In the main text, we derived the equilibrium stopping rate $\gamma(t)$ separately for the cases of a declining subsidy (equation (9)) and randomly expiring subsidy (equation (10)). It is easy to see that a policy that combines both of these elements will give rise to an equilibrium stopping rate

$$\gamma(t) = \frac{r(B_1 + s(t)) - \dot{s}(t) + \kappa(t)}{B_0 - B_1}.$$

We fix a policy $\mathbf{s} = \{s(t), \kappa(t)\}_{t=0}^{\infty}$ and derive the total welfare under it, denoted by $W^{gen}(\mathbf{s})$, as a solution to a differential equation. We proceed exactly as in the proof of Proposition 5 and use s (the “current” level of subsidy) as the state variable. As in that proof, we denote by $U(s)$ the present value of an arbitrarily picked player that remains in the game at time $t > 0$ for which $s(t) = s$. We again let $\eta(s) := -\dot{s}(s^{-1}(s))$ denote the steepness of subsidy decline at value s and we let $\kappa(s) := \kappa(s^{-1}(s))$ denote the current expiratory rate at that level (with obvious abuse of notation). By dynamic

programming, we can then decompose $U(s)$ into instant payoff and continuation value:

$$\begin{aligned}
U(s) &= 2 \frac{r(B_1 + s) + \eta(s) + \kappa(s)}{B_0 - B_1} (B - \alpha s) \cdot dt \\
&\quad + \kappa(s) \frac{2BB_1}{B_0 + B_1} \cdot dt \\
&\quad + \left(1 - 2 \frac{r(B_1 + s) + \eta(s) + \kappa(s)}{B_0 - B_1} \cdot dt - \kappa(s) \cdot dt \right) \\
&\quad \cdot e^{-r \cdot dt} (U(s) - U'(s) \eta(s) dt).
\end{aligned}$$

Rearranging, dividing by dt , and ignoring remaining terms of order dt and higher, this turns into the following differential equation:

$$\begin{aligned}
U'(s) \eta(s) &= \kappa(s) \left[\frac{2s}{B_0 - B_1} (B - \alpha s - U(s)) + \frac{2BB_1}{B_0 + B_1} - U(s) \right] \\
&\quad + 2 \frac{r(B_1 + s) + \eta(s)}{B_0 - B_1} (B - \alpha s - U(s)) - rU(s). \tag{27}
\end{aligned}$$

We again obtain a boundary condition by computing the continuation value after subsidy has either expired or declined to zero, i.e. computing (18) for $s(t) \equiv \dot{s}(t) \equiv 0$ for all t :

$$U(0) = \frac{2BB_1}{B_0 + B_1}.$$

With this boundary condition, (27) has a unique solution that we denote by $U^{gen}(s)$ for $s \in [0, s_0]$, and the total welfare is then

$$W^{gen}(\mathbf{s}) = q_0 (B - \alpha s_0) + (q_1 + q_2) U^{gen}(s_0).$$

Since we analyze an arbitrary general policy \mathbf{s} , we cannot hope to get an explicit formula for $U^{gen}(s)$. Nevertheless, by investigating (27) we can find bounds for the solution.

To analyze the range of values $U^{gen}(s)$ can take, note first that if we set $\kappa(s) \equiv 0$, the solution to (27) gives the value for some deterministically declining policy. We know from Proposition 5 that in that case the solution is bounded from above by $U_\infty^{sm}(s)$ and from below by $U^{pe}(s)$.

We now ask how adding some positive expiration rate $\kappa(s)$ affects the solution. Note that the right hand side of (27) is linear in $\kappa(s)$, and the multiplier of $\kappa(s)$ is the term in square brackets:

$$\Psi := \frac{2s}{B_0 - B_1} (B - \alpha s - U(s)) + \frac{2BB_1}{B_0 + B_1} - U(s).$$

We see that Ψ is linear and decreasing in $U(s)$. By setting $U(s) = U_\infty^{sm}(s)$, where $U_\infty^{sm}(s)$ is defined in the proof of Proposition 5, we get:

$$\Psi = \frac{\left(e^{-\frac{2s}{B_0-B_1}} + \frac{2s}{B_0-B_1}e^{-\frac{2s}{B_0-B_1}} - 1\right)(2B + (B_0 + B_1)\alpha)}{2(B_0 + B_1)} < 0.$$

(To see that this is negative, note that $1 - e^{-x} - xe^{-x} > 0$ for all $x > 0$, and hence the first term in the nominator is negative, while the other terms are positive). This shows that whenever value $U(s)$ is close to the limiting value under infinitely steeply declining deterministic policy, the effect of any positive expiration rate is strictly negative. On the other hand, setting $U(s) = U^{pe}(s)$, where U^{pe} is defined in the proof of Proposition 5, we get:

$$\Psi = \frac{2sB_1(2B + (B_0 + B_1)\alpha)}{(B_0 + B_1 + 2s)(B_0 + B_1)} > 0.$$

This shows that whenever value $U(s)$ is close to that of a subsidy that remains permanently at s , the effect of a positive expiration rate is strictly positive. Hence, the solution to (27) must lie between $U^{pe}(s)$ and $U_\infty^{sm}(s)$ for all $s > 0$. This implies that the total welfare under a general uniform policy remains within the same bounds as under an arbitrary deterministically declining subsidy policy:

$$W^{pe}(s_0) < W^{gen}(s) < W_\infty^{sm}(s_0).$$